Differential Geometry II - Smooth Manifolds
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Orientations

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1 Orientations of Vector Spaces

We begin with orientations of vector spaces. We are all familiar with certain informal rules for singling out preferred ordered bases of $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$. We usually choose a basis for $\mathbb{R}^1$ that points to the right (i.e., in the positive direction). A natural family of preferred ordered bases for $\mathbb{R}^2$ consists of those for which the rotation from the first vector to the second is in the counterclockwise direction. And every student of vector calculus encounters "right-handed" bases in $\mathbb{R}^3$: these are the ordered bases (E_1, E_2, E_3) with the property that when the fingers of your right hand curl from E_1 to E_2, your thumb points in the direction of E_3.

Although "to the right", "counterclockwise", and "right-handed" are not mathematical terms, it is easy to translate the rules for selecting preferred bases of $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$ into rigorous mathematical language: you can check that in all three cases, the preferred bases are the ones whose transition matrices from the standard basis have positive determinants.

In an abstract vector space for which there is no canonical basis, we no longer have any way to determine which bases are "correctly oriented". For example, if V is the vector space of polynomials in one real variable of degree at most 2, who is to say which of the ordered bases $(1, x, x^2)$ and $(x^2, x, 1)$ is "right-handed"? All we can say in general is what it means for two bases to have the "same orientation". Thus we are led to introduce the following definition.

Definition 1. Let V be a real vector space of dimension $n \geq 1$. We say that two ordered bases $(E_1, \ldots, E_n)$ and $(\widetilde{E}_1, \ldots, \widetilde{E}_n)$ for V are *consistently oriented* if the transition matrix $(B_i^j)_{1 \leq i,j \leq n}$, defined by

$$E_i = \sum_j B_i^j \widetilde{E}_j,$$

has positive determinant.

Exercise 2. Show that being consistently oriented is an equivalence relation on the set of all ordered bases of V, and show that there are exactly two equivalence classes.

Definition 3. Let V be a real vector space. If $\dim_{\mathbb{R}} V = n \geq 1$, we define an *orientation for V* as an equivalence class of ordered bases. If $(E_1, \ldots, E_n)$ is any ordered basis for V, we denote the orientation that it determines by $[E_1, \ldots, E_n]$, and the opposite orientation by $-[E_1, \ldots, E_n]$. On the other hand, for the special case of a zero-dimensional vector space V, we define an orientation of V to be simply a choice of one of the numbers ± 1.

Definition 4. A vector space together with a choice of orientation is called an *oriented vector space.* If V is oriented, then any ordered basis $(E_1, \ldots, E_n)$ that is in the given orientation is said to be *positively oriented* (or simply *oriented*). Any ordered basis that is not in the given orientation is said to be *negatively oriented.*

Example 5. Consider the Euclidean space $V = \mathbb{R}^n$. The orientation $[e_1, \ldots, e_n]$ of $\mathbb{R}^n$ determined by the standard basis $\{e_1, \ldots, e_n\}$ is called the *standard orientation.* You should convince yourself that, in our usual way of representing the axes graphically, an oriented basis for $\mathbb{R}^1$ is one that points to the right; an oriented basis for $\mathbb{R}^2$ is one for which the rotation from the first basis vector to the second is counterclockwise; and an oriented basis for $\mathbb{R}^3$ is a right-handed one. (These can be taken as mathematical definitions for the words “right”, “counterclockwise”, and “right-handed”.) The standard orientation for $\mathbb{R}^0$ is defined to be $+1$.

There is an important connection between orientations and alternating tensors, which is expressed in the following proposition.

Proposition 6. *Let V be a real vector space of dimension n. Each nonzero element $\omega \in \Lambda^n(V^*)$ determines an orientation $\mathcal{O}_\omega$ of V as follows: if $n \geq 1$, then $\mathcal{O}_\omega$ is the set of ordered bases $(E_1, \dots, E_n)$ for V such that $\omega(E_1, \dots, E_n) > 0$, while if $n = 0$, then $\mathcal{O}_\omega$ is $+1$ if $\omega > 0$, and -1 if $\omega < 0$. Moreover, two nonzero n-covectors on V determine the same orientation if and only if each is a positive multiple of the other.*

Proof. The 0-dimensional case is immediate, since a nonzero element of $\Lambda^0(V^*)$ is just a nonzero real number (as it is a function $\mathbb{R}^0 \to \mathbb{R}$). Thus, we may assume that $n \geq 1$. Let ω be a nonzero element of $\Lambda^n(V^*)$, and denote by $\mathcal{O}_\omega$ the set of ordered bases on which ω gives positive values. We need to show that $\mathcal{O}_\omega$ is exactly one equivalence class. Suppose (E_i) and $(\widetilde{E}_j)$ are any two ordered bases for V, and let $B \colon V \to V$ be the linear map sending E_j to $\widetilde{E}_j$ for all j. This means that the matrix of B with respect to (E_i) on the source and $(\widetilde{E}_j)$ on the target is the transition matrix between the two bases. By the [PDF: Multilinear Algebra, Proposition 21], we obtain

$$\omega(\widetilde{E}_1, \dots, \widetilde{E}_n) = \omega(BE_1, \dots, BE_n) = (\det B)\, \omega(E_1, \dots, E_n).$$

It follows that the basis $(\widetilde{E}_j)$ is consistently oriented with (E_i) if and only if $\omega(\widetilde{E}_1, \dots, \widetilde{E}_n)$ and $\omega(E_1, \dots, E_n)$ have the same sign, which is the same as saying that $\mathcal{O}_\omega$ is one equivalence class. The last statement then follows easily (and is thus left as an exercise). □

Definition 7. If V is an oriented n-dimensional real vector space and if ω is an n-covector that determines the orientation of V as described in the above proposition, then we say that ω is a *(positively) oriented n-covector*.

For example, the n-covector $\varepsilon^{1\dots n} = \varepsilon^1 \wedge \cdots \wedge \varepsilon^n$ is positively oriented for the standard orientation on $\mathbb{R}^n$.

Recall that if V is an n-dimensional real vector space, then the space $\Lambda^n(V^*)$ is 1-dimensional. Proposition 6 shows that choosing an orientation for V is equivalent to choosing one of the two components of $\Lambda^n(V^*) \setminus \{0\}$. This formulation also works for 0-dimensional vector spaces, and explains why we have defined an orientation of a 0-dimensional space in the way we did.

2 Orientations of Smooth Manifolds

In this section we briefly discuss the theory of orientations of smooth manifolds. They have numerous applications, most notably in the theory of integration on manifolds, which will be presented in *Lecture 14* of this course.

Definition 8. Let M be a smooth manifold with or without boundary. A *pointwise orientation* on M is defined to be a choice of orientation of each tangent space.

By itself, this is not a very useful concept, because the orientations at nearby points may have no relation to each other. For example, a pointwise orientation on $\mathbb{R}^n$ might switch randomly from point to point between the standard orientation and its opposite. In order for pointwise orientations to have some relationship with the smooth structure, we need an extra condition to ensure that the orientations of nearby tangent spaces are consistent with each other.

Definition 9. Let M be a smooth manifold with or without boundary, endowed with a pointwise orientation. If (E_i) is a local frame for TM over an open subset $U \subseteq M$, then we say that (E_i) is *positively oriented* (or simply *oriented*) if $(E_1|_p, \ldots, E_n|_p)$ is a positively oriented ordered basis for T_pM at each point $p \in U$; see Definition 4. A *negatively oriented* frame for TM over $U \subseteq M$ is defined analogously.

Definition 10. Let M be a smooth manifold with or without boundary.

(a) A pointwise orientation on M is said to be *continuous* if every point of M is in the domain of an oriented local frame for TM.

(b) An *orientation of M* is a continuous pointwise orientation.

(c) We say that M is *orientable* if there exists an orientation for it; otherwise we say that M is *nonorientable*.

Example 11. We give here some examples of orientable and nonorientable manifolds.

(a) Every parallelizable[1] manifold is orientable. Indeed, if $(E_1, \ldots, E_n)$ is a smooth global frame for M, then we define a pointwise orientation on M by declaring the basis $(E_1|_p, \ldots, E_n|_p)$ for T_pM to be positively oriented at each $p \in M$, and it is clear that this pointwise orientation is continuous, because every point of M is in the domain of the oriented smooth global frame (E_i). Therefore, for each $n \in \mathbb{N}$, the Euclidean space $\mathbb{R}^n$ is orientable.

(b) For each $n \in \mathbb{N}$, the unit n-sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ is orientable. Indeed, this follows from Proposition 20 (because $\mathbb{S}^n$ is a hypersurface in $\mathbb{R}^{n+1}$, to which the vector field $N = x^i\partial/\partial x^i$ is nowhere tangent) or Proposition 22 (because $\mathbb{S}^n$ is the boundary of the closed unit ball).

(c) The so-called *Möbius band* is nonorientable.

[1] A smooth manifold M with or without boundary which admits a smooth global frame or, equivalently, whose tangent bundle TM is the trivial smooth vector bundle of rank $\dim M$ (see *Exercise* 5, *Sheet* 10) is called *parallelizable*. Note that the Euclidean space $\mathbb{R}^n$ is parallelizable, and it can also be shown that $\mathbb{S}^1$, $\mathbb{S}^3$ and $\mathbb{S}^7$ are the only spheres that are parallelizable.

Definition 12. An *oriented manifold* (with or without boundary) is an orderer pair $(M, \mathcal{O})$, where M is an orientable smooth manifold (with or without boundary) and $\mathcal{O}$ is a choice of orientation for M. For each $p \in M$, the orientation of T_pM determined by $\mathcal{O}$ is denoted by $\mathcal{O}_p$.

If M is zero-dimensional, then this definition just means that an orientation of M is a choice of ± 1 attached to each of its points. The continuity condition is vacuous in this case, and the notion of oriented frames is not useful. Clearly, every 0-manifold is orientable.

Exercise 13. Let M be an oriented smooth manifold with or without boundary of dimension $n \geq 1$. Show that every local frame with connected domain is either positively oriented or negatively oriented. Moreover, show that the connectedness assumption is necessary.

2.1 Two Ways of Specifying Orientations

The following two propositions, namely Proposition 14 and Proposition 18, give ways of specifying orientations on manifolds that are more practical to use than the definition.

Proposition 14 (*The orientation determined by an n-form*)**.** *Let M be a smooth n-manifold with or without boundary. Any nonvanishing n-form ω on M determines a unique orientation of M for which ω is positively oriented at each point. Conversely, if M is given an orientation, then there is a smooth nonvanishing n-form on M that is positively oriented at each point.*

Proof. Let ω be a nonvanishing n-form on M. By Proposition 6, ω defines a pointwise orientation on M, so it remains to show that it is continuous. Since this is trivially true for $n = 0$, we may assume that $n \geq 1$. Given $p \in M$, let (E_i) be any local frame for TM over a connected open neighborhood U of p in M, and let (ε^i) be the dual coframe. On U, the expression for ω in this frame is

$$\omega = f\, \varepsilon^1 \wedge \ldots \wedge \varepsilon^n$$

for some continuous function f on U. The fact that ω is nonvanishing means that f is nonvanishing, and thus

$$\omega_p\left(E_1|_p, \ldots, E_n|_p\right) = f(p) \neq 0 \quad \text{for all } p \in U.$$

Since U is connected, it follows that this expression is either always positive or always negative on U, and therefore the given frame is either positively oriented or negatively oriented. If the latter case holds, then we can replace E_1 by $-E_1$ to obtain a new frame that is positively oriented. Hence, the pointwise orientation determined by ω is continuous.

The proof of the converse uses partitions of unity and is thus omitted. □

Due to Proposition 14, we may now give the following definition.

Definition 15. Let M be a smooth n-manifold with or without boundary. Any nonvanishing n-form on M is called an *orientation form*. If M is oriented and if ω is an orientation form determining the given orientation, then we also say that ω is *positively oriented* (or simply *oriented*).

If M is zero-dimensional, then a nonvanishing 0-form (i.e., a nonvanishing smooth real-valued function) on M assigns the orientation $+1$ to points where it is positive and -1 to points where it is negative.

Remark 16. It is easy to check that if ω and $\widetilde{\omega}$ are two positively oriented smooth n-forms on M, then $\widetilde{\omega} = f\omega$ for some strictly positive smooth real-valued function f on M; see Proposition 6.

Definition 17.

(a) A smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ for a smooth manifold M with or without boundary is said to be *consistently oriented* if for each α, β, the transition map $\varphi_\beta \circ \varphi_\alpha^{-1}$ has positive Jacobian determinant everywhere on $\varphi_\alpha(U_\alpha \cap U_\beta)$.

(b) A smooth coordinate chart $\big(U, (x^i)\big)$ on an oriented smooth manifold with or without boundary is said to be *positively oriented* (or simply *oriented*) if the coordinate frame $(\partial/\partial x^i)$ is positively oriented, and *negatively oriented* if the coordinate frame $(\partial/\partial x^i)$ is negatively oriented.

Proposition 18 (*The orientation determined by a coordinate atlas*)**.** *Let M be a smooth manifold with or without boundary of dimension $n \geq 1$. Given any consistently oriented smooth atlas for M, there exists a unique orientation for M with the property that each chart in the given atlas is positively oriented. Conversely, if M is oriented and either $\partial M = \emptyset$ or $n > 1$, then the collection of all oriented smooth charts is a consistently oriented atlas for M.*

Proof. Assume first that M has a consistently oriented smooth atlas. Each chart in the atlas determines a pointwise orientation at each point of its domain. Wherever two of the charts overlap, the transition matrix between their respective coordinate frames is the Jacobian matrix of the transition map (see the solution to part (c) of *Exercise* 1, *Sheet* 10), which has positive determinant by assumption, so they determine the same pointwise orientation at each point. The pointwise orientation on M thus determined is continuous, because each point of M is in the domain of an oriented coordinate frame.

Conversely, assume that M is oriented and either $\partial M = \emptyset$ or $n > 1$. Each point is in the domain of a smooth chart with connected domain, and if the chart is negatively oriented (see Exercise 13), then we can replace x^1 with $-x^1$ to obtain a new chart that is positively oriented. The fact that all these charts are positively oriented guarantees that their transition maps have positive Jacobian determinants, so they form a consistently oriented atlas.[2] □

Exercise 19. Let M be a connected, orientable, smooth manifold with or without boundary. Show that M has exactly two orientations. Moreover, if two orientations of M agree at one point, then they are equal.

[2] This does not work for boundary charts when $\dim M = n = 1$, because of our convention that the last coordinate is nonnegative in a boundary chart.

2.2 Orientations of Hypersurfaces

If M is an oriented smooth manifold and if S is a smooth immersed submanifold of M (with or without boundary), then S might not inherit an orientation from M, even if S is embedded. Clearly, it is not sufficient to restrict an orientation form from M to S, since the restriction of an n-form to a manifold of lower dimension must necessarily be zero. For example, the so-called *Möbius band* is nonorientable, even though it can be embedded in $\mathbb{R}^3$, which is orientable.

However, when S is an immersed or embedded *hypersurface* in M (i.e., a codimension 1-submanifold of M), it is sometimes possible to use an orientation on M to induce an orientation on S; see Proposition 20 below for the details. Recall first that a *vector field along* S is a section of the ambient tangent bundle $TM|_S$, i.e., a continuous map $N\colon S \to TM$ with the property that $N_p \in T_pM$ for every $p \in S$, and that such a vector field is said to be *nowhere tangent to* S if $N_p \in T_pM \setminus T_pS$ for all $p \in S$; cf. *Proposition 7.8*. Note that any vector field on M restricts to a vector field along S, but in general, not every vector field along S is of this form.

Proposition 20. *Let M be an oriented smooth n-manifold with or without boundary, let S be an immersed hypersurface with or without boundary in M, and let N be a vector field along S which is nowhere tangent to S. Then S has a unique orientation such that for each $p \in S$, $(E_1, \dots, E_{n-1})$ is an oriented basis for T_pS if and only if $(N_p, E_1, \dots, E_{n-1})$ is an oriented basis for T_pM.*

Note that not every hypersurface admits a nowhere tangent vector field. However, the following result gives a sufficient condition that holds in many cases.

Corollary 21. *If M is an oriented smooth manifold and if $S \subseteq M$ is a regular level set of a smooth function $f\colon M \to \mathbb{R}$, then S is orientable.*

2.3 Boundary Orientations

If M is a smooth manifold with boundary, then its boundary ∂M is an embedded hypersurface (without boundary) in M (see the *Theorem* in *Remark 9.7*(4)) and there always exists a smooth outward-pointing vector field along ∂M (see the second *Proposition* in *Remark 9.7*(5)). Since such a vector field is nowhere tangent to ∂M (see the first *Proposition* in *Remark 9.7*(5)), it determines an orientation on ∂M by Proposition 20, provided that M is oriented. The following proposition shows that this orientation is independent of the choice of an outward-pointing vector field along ∂M, and is called the *induced orientation* or the *Stokes orientation* on ∂M.

Proposition 22 (*The induced orientation on a boundary*)**.** *Let M be an oriented smooth n-manifold with boundary, where $n \geq 1$. Then ∂M is orientable, and all outward-pointing vector fields along ∂M determine the same orientation on ∂M.*

Example 23. We determine the induced orientation on $\partial\mathbb{H}^n$ when $\mathbb{H}^n$ itself has the standard orientation inherited from $\mathbb{R}^n$. We can identify $\partial\mathbb{H}^n$ with $\mathbb{R}^{n-1}$ under the correspondence

$$(x^1, \dots, x^{n-1}, 0) \leftrightarrow (x^1, \dots, x^{n-1}).$$

Since the vector field $-\partial/\partial x^n$ is outward-pointing along $\mathbb{H}^n$, the standard coordinate frame for $\mathbb{R}^{n-1}$ is positively oriented for $\partial\mathbb{H}^n$ if and only if $[-\partial/\partial x^n, \partial/\partial x^1, \ldots, \partial/\partial x^{n-1}]$ is the standard orientation for $\mathbb{R}^n$; see Proposition 20. This orientation satisfies

$$\begin{aligned}\left[-\partial/\partial x^n, \partial/\partial x^1, \ldots, \partial/\partial x^{n-1}\right] &= -\left[\partial/\partial x^n, \partial/\partial x^1, \ldots, \partial/\partial x^{n-1}\right] \\ &= (-1)^n \left[\partial/\partial x^1, \ldots, \partial/\partial x^{n-1}, \partial/\partial x^n\right].\end{aligned}$$

Thus, the induced orientation on $\partial\mathbb{H}^n$ is equal to the standard orientation on $\mathbb{R}^{n-1}$ when n is even, but it is opposite to the standard orientation when n is odd. In particular, the standard coordinates on $\partial\mathbb{H}^n \approx \mathbb{R}^{n-1}$ are positively oriented if and only if n is even.

2.4 Orientations and Smooth Maps

Definition 24. Let M and N be oriented smooth manifolds with or without boundary and let $F\colon M \to N$ be a local diffeomorphism. If both M and N are positive-dimensional, then we say that F is *orientation-preserving* if for each $p \in M$, the isomorphism dF_p takes positively oriented bases of T_pM to positively oriented bases of $T_{F(p)}N$, and *orientation-reversing* if it takes positively oriented bases of T_pM to negatively oriented bases of $T_{F(p)}N$. If, on the other hand, both M and N are zero-dimensional, then we say that F is *orientation preserving* if for every $p \in M$, the points p and $F(p)$ have the same orientation, and it is *orientation reversing* if they have opposite orientation; see the paragraph after Definition 12.

Note that a composition of orientation-preserving maps is also orientation-preserving.

Exercise 25. Let M and N be oriented positive-dimensional smooth manifolds with or without boundary and let $F\colon M \to N$ be a local diffeomorphism. Show that the following are equivalent:

(a) F is orientation-preserving.

(b) With respect to any oriented smooth charts for M and N, the Jacobian matrix of F has positive determinant.

(c) If ω is any positively oriented orientation form for N, then $F^*\omega$ is a positively oriented orientation form for M.

Proposition 26 (*The pullback orientation*)**.** *Let M and N be smooth manifolds with or without boundary. If $F\colon M \to N$ is a local diffeomorphism and if N is oriented, then M has a unique orientation, called* the pullback orientation induced by F*, such that F is orientation-preserving.*

Proof. For each $p \in M$ there is a unique orientation on T_pM that makes the isomorphism $dF_p\colon T_pM \to T_{F(p)}N$ orientation-preserving. This defines a pointwise orientation on M; provided that it is continuous, it is the unique orientation on M with respect to which F is orientation-preserving. To see that it is continuous, just choose a smooth orientation form ω of N using Proposition 14 (so that ω is positively oriented) and note that $F^*\omega$ is a smooth orientation form for M, determining by construction and by Proposition 14 the above pointwise orientation on M, which is thus continuous, as desired. □