MCAA lecture 1: quiz solutions

A) 1) X is a Markov chain

2) Y is a Markov chain

3) Z is not; see page 3

4) U is not, but see page 4

5) V is a Markov chain (actually, any (time-homogeneous) Markov chain may be written like this)

B) 1) Equivalent classes: $\{1\}, \{4\}, \{7\}, \{2,3,5,6,8,9\}$

2) Periodicity:

- periodicity is actually ill-defined for states 1, 4, 7, as the set $\{n \geq 1 : p_{ii}^{(n)} > 0\}$ is empty for $i = 1, 4, 7$; this states are called "transient" (see next time) and the periodicity of these does not matter much...
- periodicity of the equivalence class $\{2,3,5,6,8,9\}$ is 2 $(= \gcd(4,6))$

The process Z is not a Markov chain:

$$Z_1 = X_1 + X_0 \qquad Z_2 = X_2 + X_1 \qquad Z_3 = X_3 + X_2$$

$$P(Z_3 = 2 \mid Z_2 = 0, Z_1 = -2) = \frac{1}{2}$$

$$X_2 = 1 \qquad X_0 = X_1 = -1$$

$$P(Z_3 = 2 \mid Z_2 = 0, Z_1 = 2) = 0$$

$$X_2 = -1, \qquad X_0 = X_1 = 1$$

The process U is not a Markov chain, but can be transformed into a Markov chain:

$U_{n+1} = U_n + U_{n-1} + X_{n+1}$ is not Markov

$\tilde{U}_n = \begin{pmatrix} U_n \\ U_{n-1} \end{pmatrix} \in S \times S$ is Markov:

$$\tilde{U}_{n+1} = \begin{pmatrix} U_{n+1} \\ U_n \end{pmatrix} = \begin{pmatrix} U_n + U_{n-1} + X_{n+1} \\ U_n \end{pmatrix}$$

$$= \begin{pmatrix} U_n + U_{n-1} \\ U_n \end{pmatrix} + \begin{pmatrix} X_{n+1} \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} U_n \\ U_{n-1} \end{pmatrix}}_{\tilde{U}_n} + \underbrace{\begin{pmatrix} X_{n+1} \\ 0 \end{pmatrix}}_{\tilde{X}_{n+1}}$$

$\tilde{U}_{n+1} = A \cdot \tilde{U}_n + \tilde{X}_{n+1}$ (same form as Y)