Markov Chains and Algorithmic Applications: WEEK 2

1 Recurrence and transience

Definition 1.1. •

- A state $i \in S$ is **recurrent** if $f_{ii} = \mathbb{P}(\exists n \geq 1 \text{ such that } X_n = i \mid X_0 = i) = 1$
  (i.e., the probability that the chain returns to state i in finite time is equal to 1).
- A state $i \in S$ is **transient** if $f_{ii} < 1$.

So a state is recurrent if and only if it is not transient.
Note in particular that it is not necessary that $f_{ii} = 0$ for state i to be transient.

Examples.

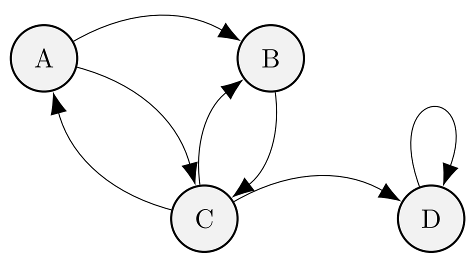


Figure 1: Here states A, B and C are transient and D is recurrent.

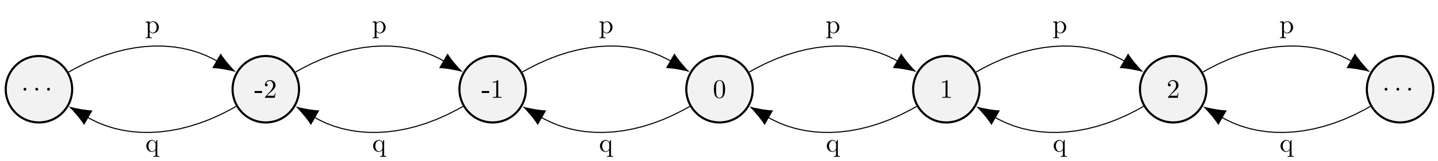


Figure 2: For $0 < p, q < 1$ and $p + q = 1$, are the states transient or recurrent ?

Facts. •

- In a given equivalence class, either all states are recurrent, or all states are transient.
- In a *finite* chain, an equivalence class is recurrent iff there is no arrow leading out of it.
  (So a finite irreducible chain is always recurrent.)
- In a infinite chain, things are more complicated. (The chain might "escape to infinity".)

In order to deal with infinite chains, we need to establish a relation between the following two sequences of numbers:

· $p_{ii}^{(n)} = \mathbb{P}(X_n = i | X_0 = i)$ (with the convention $p_{ii}^{(0)} = 1$)

· $f_{ii}^{(n)} = \mathbb{P}(X_n = i, X_{n-1} \neq i, \ldots, X_1 \neq i | X_0 = i)$ (with the convention $f_{ii}^{(0)} = 0$)

In words, $f_{ii}^{(n)}$ is the probability that, having left state i at time 0, the chain returns to state i at time n for the first time.

Lemma 1.2. $\forall n \geq 1$, we have:

$$p_{ii}^{(n)} = \sum_{m=1}^{n} f_{ii}^{(m)}\, p_{ii}^{(n-m)}$$

Proof. Let

$$\begin{aligned} A_n &= \{X_n = i\} \,:\, p_{ii}^{(n)} = \mathbb{P}(A_n | X_0 = i) \\ B_n &= \{X_n = i, X_{n-1} \neq i, \ldots, X_1 \neq i\} \,:\, f_{ii}^{(n)} = \mathbb{P}(B_n | X_0 = i) \end{aligned}$$

If the event A_n takes place, then it must be that one of the event $B_1, \ldots, B_n$ also happen (because in the worst case, X will return to state i at time n). Therefore:

$$\begin{aligned} p_{ii}^{(n)} &= \mathbb{P}(A_n | X_0 = i) = \mathbb{P}(A_n \bigcap (\textstyle\bigcup_{m=1}^{n} B_m) \,| X_0 = i) \\ &= \sum_{m=1}^{n} \mathbb{P}(A_n \cap B_m | X_0 = i) = \sum_{m=1}^{n} \mathbb{P}(A_n | B_m, X_0 = i)\, \mathbb{P}(B_m | X_0 = i) \\ &= \sum_{m=1}^{n} \underbrace{\mathbb{P}(X_n = i | X_m = i, X_{m-1} \neq i, \ldots, X_1 \neq i, X_0 = i)}_{=p_{ii}^{(n-m)}} \underbrace{\mathbb{P}(X_m = i, X_{m-1} \neq i, \ldots, X_1 \neq i | X_0 = i)}_{=f_{ii}^{(m)}} \end{aligned}$$

where we have used the Markov property in the last equality leading to the term $p_{ii}^{(n-m)}$.

□

Proposition 1.3. A state $i \in S$ is recurrent iff $\sum_{n \geq 1} p_{ii}^{(n)} = +\infty$.
(So a state $i \in S$ is transient iff $\sum_{n \geq 1} p_{ii}^{(n)} < +\infty$)

Proof. First note

$$f_{ii} = \mathbb{P}(\exists n \geq 1 \text{ s.t. } X_n = i \mid X_0 = i) = \mathbb{P}(\textstyle\bigcup_{n \geq 1} B_n | X_0 = i) = \sum_{n \geq 1} \mathbb{P}(B_n | X_0 = i) = \sum_{n \geq 1} f_{ii}^{(n)}$$

So what we need to prove is that $\sum_{n \geq 1} f_{ii}^{(n)} = 1$ iff $\sum_{n \geq 1} p_{ii}^{(n)} = +\infty$.

Observe that there is a convolution relation between $p_{ii}^{(n)}$'s and $f_{ii}^{(n)}$'s. We will therefore use generating functions to get a simpler relation. Define for $s \in [0, 1]$:

$$P_{ii}(s) = \sum_{n \geq 0} s^n p_{ii}^{(n)} \quad \text{and} \quad F_{ii}(s) = \sum_{n \geq 0} s^n f_{ii}^{(n)}$$

We will need now the following fact, also known as Abel's theorem:

Fact (Abel's Theorem). Let $(a_n, n \geq 0)$ be a sequence of numbers s.t. $0 \leq a_n \leq 1$, $\forall n \geq 0$. Then, $A(s) = \sum_{n \geq 0} s^n a_n$ converges $\forall s, |s| < 1$ and

$$\text{either } \lim_{s \to 1} A(s) = \sum_{n \geq 0} a_n \in \mathbb{R}_+ \quad \text{or } \lim_{s \to 1} A(s) = \sum_{n \geq 0} a_n = +\infty$$

So for $|s| < 1$, we have:

$$\begin{aligned} P_{ii}(s) = 1 + \sum_{n \geq 1} s^n p_{ii}^{(n)} &= 1 + \sum_{n \geq 1} s^n \left(\sum_{m=1}^{n} f_{ii}^{(m)} p_{ii}^{(n-m)} \right) \\ &= 1 + \sum_{n \geq 1} \sum_{m=1}^{n} s^m s^{n-m} f_{ii}^{(m)} p_{ii}^{(n-m)} = 1 + \sum_{m \geq 1} \sum_{n \geq m} s^m f_{ii}^{(m)} s^{n-m} p_{ii}^{(n-m)} \\ &= 1 + \sum_{m \geq 1} s^m f_{ii}^{(m)} \sum_{k \geq 0} s^k p_{ii}^{(k)} = 1 + F_{ii}(s)\, P_{ii}(s) \end{aligned}$$

remembering that $f_{ii}^{(0)} = 0$, by convention.

Hence, $P_{ii}(s) = \frac{1}{1-F_{ii}(s}$ for all $|s| < 1$ and by Abel's theorem:

$$\sum_{n\geq 0} p_{ii}^{(n)} = \lim_{s\to 1} P_{ii}(s) = +\infty \quad \text{iff} \quad f_{ii} = \sum_{n\geq 0} f_{ii}^{(n)} = \lim_{s\to 1} F_{ii}(s) = 1$$

$\square$

Remark.

$$\sum_{n\geq 1} p_{ii}^{(n)} = \sum_{n\geq 1} \mathbb{P}(X_n = i | X_0 = i) = \text{expected number of visits of state i } | X_0 = i$$

So this expected number of visits of state i is infinite iff i is recurrent.

Example 1.4. - One-dimensional simple (a-)symmetric random walk: by Homework 1, Exercise 1:

$$p_{00}^{(2n)} \approx \frac{(4pq)^n}{\sqrt{\pi n}} \quad \text{for } n \text{ large}$$

The chain is recurrent iff state 0 is recurrent iff

$$\sum_{n\geq 1} p_{00}^{(n)} = +\infty \quad \text{iff} \quad \sum_{n\geq 1} p_{00}^{(2n)} = +\infty \quad \text{iff} \quad \sum_{n\geq 1} \frac{(4pq)^n}{\sqrt{\pi n}} = \infty$$

iff $p = q = 1/2$ (else $4pq < 1$ and the series converges).

- Two-dimensional simple symmetric random walk (see Homework 1, Exercise 2):

$$p_{00}^{(2n)} \approx \frac{1}{\pi n} \quad \text{for } n \text{ large}$$

so $\sum_{n\geq 1} p_{00}^{(2n)} = +\infty$ and the chain is reccurent.

- Three-dimensional simple symmetric random walk: see Homework 2, Exercise 2.

2 Positive and null-recurrence

Let $T_i = \inf\{n \geq 1 : X_n = i\}$ be the first recurrence time to state i. So $f_{ii}^{(n)} = \mathbb{P}(T_i = n | X_0 = i)$ and

$$f_{ii} = \sum_{n\geq 1} f_{ii}^{(n)} = \sum_{n\geq 1} \mathbb{P}(T_i = n | X_0 = i) = \mathbb{P}(T_i < +\infty | X_0 = i) \begin{cases} = 1 \text{ iff i is recurrent} \\ < 1 \text{ iff i is transient} \end{cases}$$

Definition 2.1. The **mean recurrence time** to state i is defined as $\mu_i = \mathbb{E}(T_i | X_0 = i)$ $\bullet$

- if i is transient, then $\mathbb{P}(T_i = +\infty | X_0 = i) > 0$, so $\mu_i = +\infty$.
- if i is recurrent, then $\mu_i = \sum_{n\geq 1} n \, \mathbb{P}(T_i = n | X_0 = i) \geq 0 \in [1, +\infty]$.

In this case, we say that $\bullet$

- i is **positive-recurrent** if $\mu_i < +\infty$.
- i is **null-recurrent** if $\mu_i = +\infty$.

Remarks. •

- What does is mean to be recurrent ? By time-homogeneity, this implies that the chain will visit state i an infinite number of times with probability 1.

- In the case of a positive-recurrent state, the average time duration between two visits is finite.

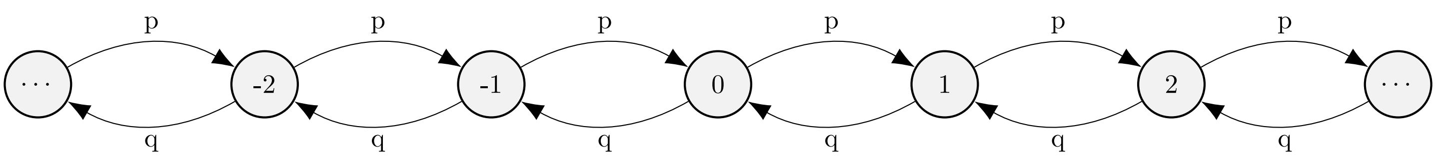
- In the case of a null-recurrent state, this average time duration between two visits is infinite, but the probability to return in finite time is 1, as counter-intuitive as it may be!

Facts. •

- In a given equivalence class, either all states are transient, or all state are positive-recurrent, or all states are null-recurrent.

- A finite irreducible chain is always positive-recurrent.

Example.



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· $p \neq q \implies$ transient chain $\implies \mathbb{P}(T_0 = +\infty | X_0 = 0) > 0$ and $\mu_0 = +\infty$

· $p = q = \frac{1}{2} \implies$ recurrent chain $\implies \mathbb{P}(T_0 = +\infty | X_0 = 0) = 0$, but $\mu_0 = +\infty$ also (without proof); the chain is null-recurrent in this second case.