Solutions 5

1. "Only if" part: Let us fix n and the sequence of $j_1, j_2, ..., j_n$. By using the detailed balance equation, we have

$$\begin{aligned}
\pi_{j_1}\, p_{j_1j_2}\, p_{j_2j_3} \cdots p_{j_{n-1}j_n}\, p_{j_nj_1} &= p_{j_2j_1}\, \pi_{j_2}\, p_{j_2j_3} \cdots p_{j_{n-1}j_n}\, p_{j_nj_1} \\
&= p_{j_2j_1}\, p_{j_3j_2} \pi_{j_3} \cdots p_{j_{n-1}j_n}\, p_{j_nj_1} \\
&= \dots \\
&= p_{j_2j_1}\, p_{j_3j_2} \cdots p_{j_nj_{n-1}}\, \pi_{j_n}\, p_{j_nj_1} \\
&= p_{j_2j_1}\, p_{j_3j_2} \cdots p_{j_nj_{n-1}}\, p_{j_1j_n}\, \pi_{j_1}
\end{aligned}$$

As we know the chain is ergodic, $\pi_{j_1} \neq 0$, so we have

$$p_{j_1j_2}\, p_{j_2j_3} \cdots p_{j_{n-1}j_n}\, p_{j_nj_1} = p_{j_2j_1}\, p_{j_3j_2} \cdots p_{j_nj_{n-1}}\, p_{j_1j_n}$$

"If" part: Let us fix n and marginalize the expression $p_{j_1j_2}\, p_{j_2j_3} \cdots p_{j_{n-1}j_n}\, p_{j_nj_1}$ over the sequences of $j_2, \dots, j_{n-1}$ as

$$\begin{aligned}
\sum_{j_2,\dots,j_{n-1}} p_{j_1j_2}\, p_{j_2j_3} \cdots p_{j_{n-1}j_n}\, p_{j_nj_1} &= p_{j_nj_1} \sum_{j_2,\dots,j_{n-1}} p_{j_1j_2}\, p_{j_2j_3} \cdots p_{j_{n-1}j_n} \\
&= p_{j_nj_1} \sum_{j_2,\dots,j_{n-1}} \mathbb{P}(X_n = j_n, X_{n-1} = j_{n-1}, \dots, X_2 = j_2 | X_1 = j_1) \\
&= p_{j_nj_1}\, \mathbb{P}(X_n = j_n | X_1 = j_1) = p_{j_nj_1}\, p_{j_1j_n}^{(n-1)}
\end{aligned}$$

By repeating the same set of calculations for the expression $p_{j_1j_n}\, p_{j_nj_{n-1}} \cdots p_{j_3j_2}\, p_{j_2j_1}$, using the assumed equality, and considering the case for which $j_1 = i$ and $j_n = k$ (i.e, fixing the first and the last state in the sequence), we have

$$p_{ki}\, p_{ik}^{(n-1)} = p_{ik}\, p_{ki}^{(n-1)}$$

Since the chain is ergodic, as n goes to $+\infty$, $p_{ik}^{(n-1)}$ and $p_{ki}^{(n-1)}$ go to π_k and π_i respectively. Therefore, we have

$$p_{ki}\pi_k = p_{ik}\pi_i$$

which is the detailed balance equation.

2. a) This chain is clearly ergodic. The transition matrix is

$$\begin{pmatrix} 1-p & p & 0 \\ 1/2 & 0 & 1/2 \\ 0 & p & 1-p \end{pmatrix}$$

Assume that the detailed balance equation is satisfied. Then

$$\pi_1^*/2 = \pi_2^* p = \pi_0^* p$$

We conclude that

$$\pi_0^* = \pi_2^* = \frac{1}{2(1+p)} \quad \pi_1^* = \frac{p}{1+p}$$

It is then easy to verify that $\pi^* = \pi^* P$, and so this is indeed a stationary distribution, which obviously satisfies the detailed balance equation.

b) We know that $\lambda_0 = 1$, and so, to compute the eigenvalues, we must solve the equations

$$\begin{aligned} 2 - 2p &= 1 + \lambda_1 + \lambda_2 \\ -p(1-p) &= \lambda_1 \lambda_2 \end{aligned}$$

Solving this, we obtain that $\lambda_1 = 1 - p$ and $\lambda_2 = -p$. So $\lambda_* = \max(p, 1-p)$ and the spectral gap is given by $\gamma = 1 - \lambda_* = \min(p, 1-p)$.

c) For $p = \frac{1}{N}$, the spectral gap is $\gamma = \frac{1}{N}$. From the theorem seen in class, we know that $\|P_i^n - \pi\|_{\mathrm{TV}} \leq \frac{\exp(-\gamma n)}{2\sqrt{\pi_i}}$, so here,

$$\max_{i \in S} \|P_i^n - \pi\|_{\mathrm{TV}} \leq \frac{1}{2} \sqrt{\frac{1 + 1/N}{1/N}} \exp(-n/N) \leq \sqrt{N} \exp(-n/N) = \exp\left(\frac{\log N}{2} - \frac{n}{N}\right)$$

Taking therefore $n \geq N\left(\frac{\log N}{2} + c\right)$ with $c > 0$ sufficiently large (more precisely, $c = \log(1/\varepsilon)$) ensures that the maximum total variation norm is below ε.

d) For $p = 1 - \frac{1}{N}$, the spectral gap is again $\gamma = \frac{1}{N}$. So

$$\max_{i \in S} \|P_i^n - \pi\|_{\mathrm{TV}} \leq \frac{1}{2} \sqrt{2(2 - 1/N)} \exp(-n/N) \leq \exp(-n/N)$$

Taking therefore $n \geq cN$ with $c = \log(1/\varepsilon)$ ensures that the maximum total variation norm is below ε, so

$$T_\varepsilon \leq N \log(1/\varepsilon)$$

3. a) The transition matrix being doubly stochastic, the stationary distribution is uniform (i.e. $\pi_i = \frac{1}{2N}$ for every $i \in S$) and satisfies the detailed balance equation.

b) Solving the equation $P\phi^{(1)} = \lambda\phi^{(1)}$, we obtain

$$\begin{aligned} \tfrac{N-1}{N}\, a + \tfrac{1}{N}\, b &= \lambda\, a \\ \tfrac{N-1}{N}\, a - \tfrac{1}{N}\, b &= \lambda\, b \end{aligned}$$

which is saying that λ is an eigenvalue of the 2×2 matrix

$$\begin{pmatrix} \frac{N-1}{N} & \frac{1}{N} \\ \frac{N-1}{N} & -\frac{1}{N} \end{pmatrix} = \begin{pmatrix} 1-\delta & \delta \\ 1-\delta & -\delta \end{pmatrix}$$

where we have set $\delta = \frac{1}{N}$. These eigenvalues are given by

$$\lambda_\pm = \frac{1 - 2\delta \pm \sqrt{(1-2\delta)^2 + 8\delta(1-\delta)}}{2} = \frac{1 - 2\delta \pm \sqrt{1 + 4\delta - 4\delta^2}}{2}$$

For δ small (i.e. N large), the largest of these 2 eigenvalues is λ_+, which is approximately given by

$$\lambda_+ \simeq \frac{1 - 2\delta + (1 + 2\delta - 4\delta^2)}{2} = 1 - 2\delta^2 = 1 - \frac{2}{N^2}$$

so the spectral gap $\gamma \simeq \frac{2}{N^2}$.

c) By the theorem seen in class,

$$\max_{i \in S} \|P_i^n - \pi\|_{\text{TV}} \le \frac{\sqrt{2N}}{2}\, \exp(-\gamma n) \le \sqrt{2} \exp\left(\frac{\log N}{2} - \frac{2n}{N^2}\right)$$

is below ε for $n \ge \frac{N^2}{2}\left(\frac{\log N}{2} + \log(\sqrt{2}/\varepsilon)\right)$, so

$$T_\varepsilon \le \frac{N^2}{2}\left(\frac{\log N}{2} + \log(\sqrt{2}/\varepsilon)\right)$$