Homework 2

Exercise 1. Let $(X_n,\, n \geq 1)$ be a sequence of i.i.d. random variables such that $\mathbb{P}(X_n = +1) = \mathbb{P}(X_n = -1) = \frac{1}{2}$ for every $n \geq 1$. Let also $(S_n,\, n \geq 0)$ the simple symmetric random walk defined as $S_0 = 0$, $S_{n+1} = S_n + X_{n+1}$ for $n \geq 0$.

Among the following discrete-time stochastic processes, which are (time-homogeneous) Markov chains, which are not? If a process is a Markov chain, prove it *and* compute its transition matrix P [1]. If a process is not, find a counter-example showing that the Markov property is not satisfied (*Hint:* Consider small values of n !)

a) $Y_n = S_{2n}$ for $n \geq 0$.

b) $Z_n = (-1)^{S_n}$ for $n \geq 0$.

c) $T_n = \max\{S_0, S_1, \ldots, S_n\}$ for $n \geq 0$.

d) $W_0 = 0$ and $W_{n+1} = W_n + X_{2n+1} + X_{2n+2}$ for $n \geq 0$.

Exercise 2. Let $(X_n,\, n \geq 0)$ be a time-homogeneous Markov chain with state space $\mathcal{S} = \{0, 1\}$ and transition probabilities

$$\mathbb{P}(X_{n+1} = 1 \,|\, X_n = 0) = p \quad \text{and} \quad \mathbb{P}(X_{n+1} = 0 \,|\, X_n = 1) = q$$

where $0 < p, q < 1$.

a) Write down the transition matrix P of the chain and draw its transition graph.

b) For which values of p, q does it hold that $(X_n,\, n \geq 0)$ is a sequence of independent random variables?

c) Compute the n-step transition probabilities $p_{00}^{(n)} = \mathbb{P}(X_n = 0 \,|\, X_0 = 0)$.

Hint: Use the eigenvalue-eigenvector decomposition of the matrix P.

d) Compute $\sum_{n \geq 1} p_{00}^{(n)}$. Is it finite or not? What does that imply?

e) Let now $T_0 = \inf\{n \geq 1 \,:\, X_n = 0\}$. Compute $f_{00}^{(n)} = \mathbb{P}(T_0 = n \,|\, X_0 = 0)$ and $f_{00} = \mathbb{P}(T_0 < +\infty \,|\, X_0 = 0)$. Is your result coherent with what you have obtained in question d)?

f) Compute $\mathbb{E}(T_0 \,|\, X_0 = 0)$. Is it finite or not? What does that imply?

g) For questions e) and f), consider the following special cases: 1) $p + q = 1$ and 2) $p = q$. Interpret your results.

[1] *NB:* Computing the transition matrix P does not prove by any means that the process is a Markov process!

Exercise 3. Let $(X_n,\, n \in \mathbb{N})$ be a time-homogeneous Markov chain with n-step transition probabilities

$$p_{ij}^{(n)} = \mathbb{P}(X_n = j \,|\, X_0 = i)$$

a) Using the criterion for recurrence seen in the lectures, show that in a given equivalence class, either all states are recurrent, or all states are transient.

We define now the probability of *first passage* as the probability that the chain passes from i to j in n steps without passing by j before the n^{th} step:

$$f_{ij}^{(n)} = \mathbb{P}(X_n = j,\, X_{n-1} \neq j, \dots,\, X_1 \neq j \,|\, X_0 = i)$$

Note: When $i = j$, this matches the definition of $f_{ii}^{(n)}$ seen in class. Let also

$$P_{ij}(s) = \sum_{n=0}^{\infty} p_{ij}^{(n)} s^n \qquad p_{ij}(0) = \delta_{ij}$$
$$F_{ij}(s) = \sum_{n=0}^{\infty} f_{ij}^{(n)} s^n \qquad f_{ij}(0) = 0$$

be the associated generating functions. Recall that we proved in class that $P_{ii}(s) = 1 + F_{ii}(s)\, P_{ii}(s)$.

b) Prove that for $i \neq j$: $P_{ij}(s) = F_{ij}(s)\, P_{jj}(s)$

c) Deduce the following statements:

1. If j is recurrent, then $\sum_{n\geq 0} p_{ij}^{(n)} = +\infty$ for all i such that $f_{ij} > 0$, where $f_{ij} = \sum_{n\geq 0} f_{ij}^{(n)}$.
2. If j is transient, then $\sum_{n\geq 0} p_{ij}^{(n)} < +\infty$ for all i.