Solutions 1

1. In order to compute $p_{00}^{(2n)}$, you should realize the following:

(i) the random walk must go up as many times as it goes down in order to come back to 0 in $2n$ steps: it therefore goes up n times and goes down n times;

(ii) the number of possible paths is equal to the number of choices for the n "up" moves among the $2n$ time instants available: this number is $\binom{2n}{n}$;

(iii) the probability of each path is the same: $p^n\, q^n$, as each path goes up n times and goes down n times.

In total, the probability is therefore given by

$$p_{00}^{(2n)} = \binom{2n}{n} p^n\, q^n$$

Using then Stirling's approximation for $\binom{2n}{n} = \frac{2n!}{n!n!}$, we obtain

$$\binom{2n}{n} p^n\, q^n \sim \frac{\sqrt{2\pi(2n)}\, \left(\frac{2n}{\mathrm{e}}\right)^{2n}}{2\pi n\, \left(\frac{n}{\mathrm{e}}\right)^{2n}} (pq)^n = \frac{(4pq)^n}{\sqrt{\pi n}}$$

2. a) Both X and Y are random walks with probability $1/4$ to go in either direction, and probability $1/2$ to stay in place.

b) No, they are not independent: when X makes a move, Y does not, and vice-versa.

c) Both U and V are simple symmetric random walks with probability $1/2$ to go in either direction.

d) Yes, they are independent. Denote $U_n = \eta_1 + \ldots + \eta_n$, $V_n = \chi_1 + \ldots + \chi_n$. Then one can check e.g. that (and similarly for all ± 1 combinations)

$$\mathbb{P}(\eta_n = +1, \chi_n = +1) = \mathbb{P}\left(\overrightarrow{\xi_n} = (+1, 0)\right) = \frac{1}{4} = \mathbb{P}(\eta_n = +1) \cdot \mathbb{P}(\chi_n = +1)$$

e) Note that $\overrightarrow{S_{2n}} = (0,0)$ if and only if $U_{2n} = V_{2n} = 0$, so by the independence shown above, we obtain

$$\begin{aligned}\mathbb{P}\left(\overrightarrow{S_{2n}} = (0,0) \,|\, \overrightarrow{S_0} = (0,0)\right) &= \mathbb{P}(U_{2n} = 0,\, V_{2n} = 0 \,|\, U_0 = 0,\, V_0 = 0) \\ &= \mathbb{P}(U_{2n} = 0 \,|\, U_0 = 0) \cdot \mathbb{P}(V_{2n} = 0 \,|\, V_0 = 0) = \left(\binom{2n}{n} 2^{-2n}\right)^2 \sim \frac{1}{\pi n}\end{aligned}$$

by Exercise 1.

3. Consider i and j are two intercommunicating states. For arbitrary m, n, and $r \in \mathbb{N}$, we have

$$\begin{aligned} p_{ii}^{(m+n+r)} &= \mathbb{P}(X_{m+n+r} = i | X_0 = i) = \sum_{k_1,k_2} \mathbb{P}(X_{m+n+r} = i, X_{m+r} = k_2, X_m = k_1 | X_0 = i) \\ &= \sum_{k_1,k_2} \mathbb{P}(X_{m+n+r} = i | X_{m+r} = k_2)\, \mathbb{P}(X_{m+r} = k_2 | X_m = k_1)\, \mathbb{P}(X_m = k_1 | X_0 = i) \end{aligned}$$

which can be rewritten as

$$p_{ii}^{(m+n+r)} = \sum_{k_1,k_2} p_{k_2 i}^{(n)}\, p_{k_1 k_2}^{(r)}\, p_{i k_1}^{(m)} \geq p_{ji}^{(n)}\, p_{jj}^{(r)}\, p_{ij}^{(m)}$$

Since i and j are intercommunicating states, there always exist m and $n \in \mathbb{N}$ such that $p_{ij}^{(m)} > 0$ and $p_{ji}^{(n)} > 0$. So, let us consider n and m fixed, and define $\alpha = p_{ji}^{(n)} p_{ij}^{(m)} > 0$. The inequality then can be rewritten as a function of α:

$$p_{ii}^{(m+n+r)} \geq \alpha\, p_{jj}^{(r)}$$

Therefore, $p_{jj}^{(r)}$ can be non-zero only if $p_{ii}^{(m+n+r)}$ is non-zero. $p_{ii}^{(m+n+r)}$ is non-zero only if $d(i) | m + n + r$. At the same time, for the case $r = 0$, we have $p_{ii}^{(m+n)} \geq \alpha > 0$, which means that $d(i) | m + n$. Therefore, $p_{jj}^{(r)}$ can be non-zero only if $d(i) | r$, which means that $d(i) | d(j)$. With the same argument, we have $d(j) | d(i)$, and as a conclusion we have $d(j) = d(i)$.