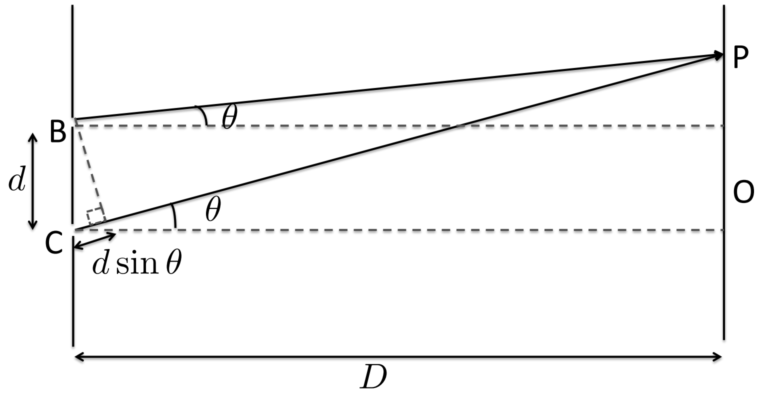
Homework 1 Solution
Traitement Quantique de l'Information

Exercise 1 *The Young double slit experiment (1803)*

1) The scheme of the experiment is as follows:



If $D >> d$, we use the approximation

$$|\psi(\vec{r}_P)|^2 \approx \frac{A^2}{D^2}\left|e^{\frac{2\pi i}{\lambda}|\vec{r}_B-\vec{r}_P|} + e^{\frac{2\pi i}{\lambda}|\vec{r}_C-\vec{r}_P|}\right|^2.$$

By factoring out the factor whose modulus is 1, we then have

$$|\psi(\vec{r}_P)|^2 \approx \frac{A^2}{D^2}\left|1 + e^{\frac{2\pi i}{\lambda}(|\vec{r}_C-\vec{r}_P|-|\vec{r}_B-\vec{r}_P|)}\right|^2.$$

As shown in the above figure, the difference of lengths between the two beams $|\vec{r}_C - \vec{r}_P| - |\vec{r}_B - \vec{r}_P|$ is $d\sin\theta$. Therefore, we have

$$\begin{aligned}
|\psi(\vec{r}_P)|^2 \approx \frac{A^2}{D^2}\left|1 + e^{\frac{2\pi i d\sin\theta}{\lambda}}\right|^2 &= \frac{A^2}{D^2}\left[\left(1+\cos\left(\frac{2\pi d\sin\theta}{\lambda}\right)\right)^2 + \sin^2\left(\frac{2\pi d\sin\theta}{\lambda}\right)\right] \\
&= \frac{A^2}{D^2}\left[2 + 2\cos\left(\frac{2\pi d\sin\theta}{\lambda}\right)\right] \\
&= \frac{4A^2}{D^2}\cos^2\left(\frac{\pi d}{\lambda}\sin\theta\right),
\end{aligned}$$

where the last line uses $\cos 2\alpha = 2\cos^2\alpha - 1$.

Note: We used an approximation which was deduced with a geometric argument, but the same results can also be obtained algebraically. For instance, let $M = \frac{1}{2}(B + C)$ be the middle of the segment $[BC]$ and ϕ the direct angle between $\vec{MO}$ and $\vec{MP}$.

$$\begin{aligned}
|\vec{r}_P - \vec{r}_B| = \left\|\vec{BP}\right\| &= \left\|\vec{BM} + \vec{MP}\right\| \\
&= \sqrt{BM^2 + MP^2 - 2\vec{MP}\cdot\vec{MB}} \\
&= MP\sqrt{1 - 2\frac{\vec{MP}\cdot\vec{MB}}{MP^2} + \left(\frac{MB}{MP}\right)^2} \\
&\simeq MP\left(1 - \frac{\vec{MP}\cdot\vec{MB}}{MP^2} + \frac{1}{2}\left(\frac{MB}{MP}\right)^2 + o\left(\frac{d^2}{D^2}\right)\right)
\end{aligned}$$

where we used $\frac{\vec{AP}\cdot\vec{AB}}{AP^2} \sim \frac{d\rho}{D^2}$ and $\frac{\vec{AP}\cdot\vec{AB}}{AP^2} \sim \left(\frac{d}{D}\right)^2$. Similarly:

$$CP \simeq MP\left(1 - \frac{\vec{MP}\cdot\vec{MC}}{MP^2} + \frac{1}{2}\left(\frac{MC}{MP}\right)^2 + o\left(\frac{d^2}{D^2}\right)\right) \tag{1}$$

Therefore, using $MB = MC$ we have:

$$CP - BP \simeq \frac{\vec{MP}}{MP}\cdot\left(\vec{MB} - \vec{MC}\right) = \frac{\vec{MP}}{MP}\cdot\vec{CB} = d\sin(\phi) \tag{2}$$

and notice that for large D compared to d and ρ, the geometric angle θ can be assimilated with ϕ with $\phi \simeq \theta$.

2) The intensity attains its minima at 0 when $\sin\theta = \left(m + \frac{1}{2}\right)\frac{\lambda}{d}$. The intensity attains its maxima when the cosine function equals ± 1, whereby $\sin\theta = m\frac{\lambda}{d}$ for some integer m.

3) For $D >> d$, we use the approximation $\tan\theta \approx \theta \approx \sin\theta$ so that the intensity is given by

$$|\psi(\vec{r}_P)|^2 \approx \frac{4A^2}{D^2}\cos^2\left(\frac{\pi d\rho}{D\lambda}\right).$$

As the location of maxima satisfies $\frac{d\rho_m}{D\lambda} = m \in \mathbb{N}$, the distance between two successive minima is

$$\rho_{m+1} - \rho_m = \lambda\frac{D}{d}.$$

With $d = 0.25$mm, $D = 10$m and $\lambda = 652$nm, the $\rho_{m+1} - \rho_m$ is 26.1mm.

Exercise 2 *Modern Young's experiment*

1) For a molecule of C_{60} $p = mv$, where $m = \frac{M_{\text{mole}}}{N_A} \times 60$. The De Broglie wavelength is $\lambda = \frac{h}{p} = \frac{hN_A}{60 M_{\text{mole}} v}$. For an average velocity of 220m/s, the wavelength is 2.518×10^{-12}m.

2) Take the results known for waves. We should observe interference fringes with a distance $\rho_{m+1} - \rho_m = \lambda \frac{D}{d} = 31.48\ \mu$m.

3) The wavelength is 5.30×10^{-35}m, which is not a measurable distance.

Exercise 3 *Photoelectric effect*

According to Einstein's formula, the kinetic energy of the ejected electrons is

$$\frac{1}{2}mv^2 = h\nu - W_0$$

where $W_0 = h\nu_0 = \frac{hc}{\lambda_0}$ is the minimal energy for extraction. The equation can be rewritten as $\frac{hc}{\lambda} = \frac{1}{2}mv^2 + \frac{hc}{\lambda_0}$. Therefore, the necessary wavelength is

$$\lambda = \left(\frac{1}{2}mv^2 + \frac{hc}{\lambda_0}\right)^{-1} hc.$$

Numerics can be calculated using $\frac{1}{2}mv^2 = 1.5$eV and one finds $\lambda = 180$nm.