Problem Set 3

For the Exercise Sessions on Oct 10 and Oct 17

Last name	First name	SCIPER Nr	Points

Problem 1: Some review problems on linear algebra

(a) (Frobenius norm) Prove that $\|A\|_F^2 = \text{trace}(A^H A)$.

(b) (Singular Value Decomposition) Let $\sigma_i(A)$ denote the i^{th} singular value of an $m \times n$ matrix A. Prove that $\|A\|_F^2 = \sum_{i=1}^{\min\{m,n\}} \sigma_i^2(A)$

(c) (Projection Matrices) Consider a set of k orthonormal vectors in $\mathbb{C}^n$, denoted by $\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_k}$. The projection matrix (that projects an arbitrary vector into the subspace spanned by these orthonormal vectors) is given by

$$P = \sum_{i=1}^{k} \mathbf{u}_i \mathbf{u}_i^H. \tag{1}$$

- Prove that this matrix is *Hermitian,* i.e., $P^H = P$.
- Prove that this matrix is *idempotent,* i.e., $P^2 = P$. (In words, projecting twice into the same subspace is the same as projecting only once.)
- Prove that $\text{trace}(P) = k$, i.e., equal to the dimension of the subspace.
- Prove that the diagonal entries of P must be real-valued and non-negative. Then, prove that the diagonal entries of P cannot be larger than 1 (this is a little more tricky).

Problem 2: Eckart–Young Theorem

In class, we proved the converse part of the Eckart–Young theorem for the spectral norm. Here, you do the same for the case of the Frobenius norm.

(a) For any matrix A of dimension $m \times n$ and an arbitrary orthonormal basis $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ of $\mathbb{C}^n$, prove that

$$\|A\|_F^2 = \sum_{k=1}^{n} \|A\mathbf{x}_k\|^2. \tag{2}$$

(b) Consider any $m \times n$ matrix B with $\text{rank}(B) \leq p$. Clearly, its null space has dimension no smaller than $n - p$. Therefore, we can find an orthonormal set $\{\mathbf{x}_1, \cdots, \mathbf{x}_{n-p}\}$ in the null space of B. Prove that for such vectors, we have

$$\|A - B\|_F^2 \geq \sum_{k=1}^{n-p} \|A\mathbf{x}_k\|^2. \tag{3}$$

(c) (This requires slightly more subtle manipulations.) For any matrix A of dimension $m \times n$ and any *orthonormal* set of $n-p$ vectors in $\mathbb{C}^n$, denoted by $\{\mathbf{x}_1, \cdots, \mathbf{x}_{n-p}\}$, prove that

$$\sum_{k=1}^{n-p} \|A\mathbf{x}_k\|^2 \quad \geq \quad \sum_{j=p+1}^{r} \sigma_j^2. \tag{4}$$

Hint: Consider the case $m \geq n$ and the set of vectors $\{\mathbf{z}_1, \cdots, \mathbf{z}_{n-p}\}$, where $\mathbf{z}_k = V^H \mathbf{x}_k$. Express your formulas in terms of these and the SVD representation $A = U\Sigma V^H$.

(d) Briefly explain how *(a)-(c)* imply the desired statement.

Problem 3: A Hilbert space of matrices

In this problem, we consider the set of matrices $A \in \mathbb{R}^{m\times n}$ with standard matrix addition and multiplication by scalar.

(a) Briefly argue that this is indeed a vector space, using the definition given in class.

(b) Show that $\langle A, B\rangle = \text{trace}(B^H A)$ is a valid inner product.

(c) Explicitly state the norm induced by this inner product. Is this a norm that you have encountered before?

(d) Consider as a further inner product candidate the form $\langle A, B\rangle = \text{trace}(B^H W A)$, where W is a square ($m \times m$) matrix. Give conditions on W such that this is a valid inner product. Explicit and detailed arguments are required for full credit.

Problem 4: Haar Wavelet

This problem is taken from Vetterli/Kovacevic, p. 295.

Consider the wavelet series expansion of continuous-time signals $f(t)$ and assume that $\psi(t)$ is the Haar wavelet.

(a) Give the expansion coefficients for $f(t) = 1, t \in [0,1]$, and 0 otherwise.

(b) Verify that for $f(t)$ as in Part *(a)*, $\sum_m \sum_n \|\langle \psi_{m,n}, f\rangle\|^2 = 1$ (i.e., Parseval's identity).

Problem 5: Dual Representation of Norm

(a) Assume that $p > 0$ and $q > 0$ fulfils $1/p + 1/q = 1$- Show that the following inequality holds for all $a \geq 0$ and $b \geq 0$.

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{5}$$

Show that the equality holds if $a^p = b^q$. [Hint: Use the concavity of log function]

(b) Given vectors $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$ show that,

$$\frac{\sum_{i=1}^n |x_i y_i|}{||\mathbf{x}||_p ||\mathbf{y}||_q} \leq 1 \tag{6}$$

What is the condition for equality?

(c) Show that

$$||\mathbf{x}||_p = sup< \mathbf{y}, \mathbf{x} >: \mathbf{y} \in \mathbb{R}^n, ||\mathbf{y}||_{p^*} = 1.$$

where $1/p + 1/p^* = 1$

Problem 6: Finding the Fair Coin

To be done in the exercise session on Oct 17.

Your colleague challenges you to a game. It goes as follows: Your are given three coins, one being fair, two being weighted. Coin one has a probability of flipping heads $1/2$, coin two has probability of flipping heads $1/2-p$, and coin three has probability of flipping heads $1/2+p$. You can assume that $p \in (0, 1/3]$. You are allowed to flip every coin m times. If thereafter, you manage to correctly identify the fair coin, you win.Otherwise you lose.

a) Describe a simple strategy that is winning for $m \to \infty$.

b) Write down the events that make this strategy fail (assume finite m).

c) Give a simple upper bound on the total failure probability in terms of m and p.

d) Your colleague makes the following proposal: you can flip each coin $m = 20$ times, and he ensures you that $p = 1/3$ (he is a good friend, so you trust him). If you can identify the fair coin, you win 3 CHF, otherwise you lose 2 CHF. Do you accept?

In general, for $p \in (0, 1/3]$ fixed, for which values of m, α would you agree to play the game, where α is defined as the ratio between the potential gain and potential loss in CHF? You can assume that you know p. It is sufficient to state bounds that are tight up to multiplicative constants.