Solutions 12

1. For this exercise, let $(U_n,\, n \geq 1)$ be a sequence of i.i.d.$\sim \mathcal{U}([0,1])$ random variables.

First case. $X_0 = 0$, $Y_0 = 1$.

a) One coupling that maximizes the chances of X and Y to meet after the first step is described as follows:

$$\begin{cases} \text{if } 0 \leq U_{n+1} \leq \frac{1}{4} & \text{then } X_{n+1} = X_n + 1 \text{ and } Y_{n+1} = Y_n \\ \text{if } \frac{1}{4} < U_{n+1} \leq \frac{1}{2} & \text{then } X_{n+1} = X_n \text{ and } Y_{n+1} = Y_n - 1 \\ \text{if } \frac{1}{2} < U_{n+1} \leq \frac{3}{4} & \text{then } X_{n+1} = X_n \text{ and } Y_{n+1} = Y_n \\ \text{if } \frac{3}{4} < U_{n+1} \leq 1 & \text{then } X_{n+1} = X_n - 1 \text{ and } Y_{n+1} = Y_n + 1 \end{cases}$$

With this coupling, the probability that X and Y meet after one step is $\frac{1}{2}$, which can be seen to be the maximum.

b) Let ξ_{n+1} be the random variable defined as

$$\xi_{n+1} = \begin{cases} +1 & \text{if } 0 \leq U_{n+1} \leq \frac{1}{4} \\ 0 & \text{if } \frac{1}{4} < U_{n+1} \leq \frac{3}{4} \\ -1 & \text{if } \frac{3}{4} < U_{n+1} \leq 1 \end{cases}$$

If both $X_{n+1} = X_n + \xi_{n+1}$ and $Y_{n+1} = Y_n + \xi_{n+1}$, then the two chains never meet.

But another option is also to have $X_{n+1} = X_n + \xi_{n+1}$ and $Y_{n+1} = Y_n - \xi_{n+1}$.

Variant: $X_0 = 0$, $Y_0 = 2$.

a) In this case, one coupling that maximizes the chances of X and Y to meet after the first step is:

$$\begin{cases} \text{if } 0 \leq U_{n+1} \leq \frac{1}{4} & \text{then } X_{n+1} = X_n + 1 \text{ and } Y_{n+1} = Y_n - 1 \\ \text{if } \frac{1}{4} < U_{n+1} \leq \frac{3}{4} & \text{then } X_{n+1} = X_n \text{ and } Y_{n+1} = Y_n \\ \text{if } \frac{3}{4} < U_{n+1} \leq 1 & \text{then } X_{n+1} = X_n - 1 \text{ and } Y_{n+1} = Y_n + 1 \end{cases}$$

With this coupling, the probability that X and Y meet after one step is $\frac{1}{4}$, which can be seen to be the maximum (*NB:* This coupling can also be described with the random variable ξ_{n+1} above: $X_{n+1} = X_n + \xi_{n+1}$ and $Y_{n+1} = Y_n - \xi_{n+1}$).

b) In this case, only the coupling $X_{n+1} = X_n + \xi_{n+1}$ and $Y_{n+1} = Y_n + \xi_{n+1}$ ensures that the walks never meet. There is no other coupling guaranteeing this property.