Homework 10

Exercise 1. a) Let d be a positive integer and $S = \{0,1\}^d$ and let $f : S \to \mathbb{R}$ be an arbitrary function. Explain how to find a reasonable approximation of the minimum of this function using the Metropolis algorithm.

Hint: Start with the simple and symmetric random walk on the hypercube S (cf. Lecture 7).

b) In the particular case where $f(x) = |x|$, where $|x|$ is the Hamming weight of $x \in S$ (that is, the number of 1's in the vector x), compute explicitly the transition matrix of the Metropolis chain, not forgetting to compute p_{xx} for $x \in S$.

c) Compute also explicitly the corresponding stationary distribution π_β of the Metropolis chain (including the normalization constant Z_β), for a fixed value of $\beta > 0$.

NB: Parts b) and c) were asked in the 2022-2023 final exam.

Exercise 2. On the state space $S = \{0,1,2\}$ and given $\beta > 0$, consider the following distribution:

$$\pi = \frac{1}{Z}\left(1, e^{-2\beta}, e^{-\beta}\right)$$

where the normalization constant $Z = 1 + e^{-2\beta} + e^{-\beta}$ is easy to compute in this case. For any given $\beta > 0$, we would like to sample from π, in order to obtain (by taking β large) an estimate of the global minimum of the function $f : S \to \mathbb{Z}$ defined as $f(0) = 0$, $f(1) = 2$ and $f(2) = 1$. Of course, in this situation, both finding the global minimum of f and sampling from the distribution π are trivial tasks, but the idea here is to get an idea of the performance (i.e. rate of convergence) of the Metropolis-Hastings algorithm in a simple case.

Consider the base chain on S with transition probabilities

$$\psi_{01} = \psi_{21} = 1 \quad \text{and} \quad \psi_{10} = \psi_{12} = \frac{1}{2}.$$

a) Compute the transition probabilities p_{ij} of the corresponding Metropolis chain.

b) Check that the detailed balance equation is satisfied.

c) Compute the eigenvalues $\lambda_0 \geq \lambda_1 \geq \lambda_2$ of P. (*Hint:* You already know that $\lambda_0 = 1$.)

d) Express the spectral gap γ as a function of β. How does it behave as β gets large?