Markov Chains and Algorithmic Applications - IC - EPFL

Homework 9

Exercise 1. [Barker's algorithm]

Let $\pi = (\pi_i,\, i \in S)$ be a distribution on a finite state space S such that $\pi_i > 0$ for all $i \in S$ and let us consider the base chain with transition probabilities ψ_{ij}, which is assumed to be irreducible, aperiodic and such that $\psi_{ij} > 0$ if and only if $\psi_{ji} > 0$. Define the following acceptance probabilities:

$$a_{ij} = \frac{\pi_j\, \psi_{ji}}{\pi_i\, \psi_{ij} + \pi_j\, \psi_{ji}}$$

as well as a new chain with transition probabilities $p_{ij} = \psi_{ij}\, a_{ij}$ if $j \neq i$. Show that this new chain is ergodic and that it satisfies the detailed balance equation:

$$\pi_i\, p_{ij} = \pi_j\, p_{ji}, \quad \forall i, j \in S$$

Exercise 2. Let π be a probability distribution $S = \mathbb{N}^* = \{1, 2, 3, \ldots\}$. We assume that $\pi_i > 0$ for every $i \in S$ and moreover that $\pi_i \geq \pi_{i+1}$ for every $i \in S$. In order to sample from π, let us consider the base chain with transition probabilities:

$$\psi_{1,2} = 1, \quad \psi_{i,i\pm1} = \frac{1}{2}, \quad \text{for } i \geq 2$$

and $\psi_{ij} = 0$ for all other values of i, j (*NB:* Does this chain satisfy the required assumptions?).

a) Compute the general expression for the acceptance probabilities a_{ij} and the transition probabilities p_{ij} of the corresponding Metropolis chain.

b) Consider then the following three particular cases (where the constants C_1, C_2, C_3 are appropriate normalization constants):

1. $\pi_i = C_1/i^2$, $i \geq 1$
2. $\pi_i = C_2 \exp(-i)$, $i \geq 1$
3. $\pi_i = C_3 \exp(-i^2)$, $i \geq 1$

In each case, compute the acceptance probabilities a_{ij}, as well as the limit $\lim_{i\to\infty} a_{i,i+1}$.

Exercise 3. Let $n \geq 1$, $0 < p < 1$, and consider the binomial distribution on $S = \{0, 1, \ldots, n\}$ defined as

$$\pi_k = \binom{n}{k} p^k (1-p)^{n-k}, \quad \text{for } k \in S$$

Construct a base chain on S, as well as the corresponding Metropolis chain whose stationary and limiting distribution is π (simplifying as much as you can the expression for the acceptance probabilities).

Exercise 4. [Metropolized independent sampling in a particular case]

Let $0 < \theta < 1$ and let us consider the following distribution π on $S = \{1, \ldots, N\}$:

$$\pi_i = \frac{1}{Z}\, \theta^{i-1}, \quad i = 1, \ldots, N$$

where Z is the normalization constant, whose computation is left to the reader.

a) Consider the base chain $\psi_{ij} = \frac{1}{N}$ for all $i, j \in S$ and derive the transition probabilities p_{ij} obtained with the Metropolis-Hastings algorithm.

b) Using the result of the course, derive an upper bound on $\|P_i^n - \pi\|_{\mathrm{TV}}$. Compare the bounds obtained for $i = 1$ and $i = N$ (for large values of N).

c) Deduce an upper bound on the (order of magnitude of the) mixing time

$$T_\varepsilon = \inf\{n \geq 1 \,:\, \max_{i \in S} \|P_i^n - \pi\|_{\mathrm{TV}} \leq \varepsilon\}$$