Homework 2

Exercise 1. a) Which of the following are cdfs?

1. $F_1(t) = \exp(-e^{-t})$, $t \in \mathbb{R}$
2. $F_2(t) = \frac{1}{1-e^{-t}}$, $t \in \mathbb{R}$
3. $F_3(t) = 1 - \exp(-1/|t|)$, $t \in \mathbb{R}$
4. $F_4(t) = 1 - \exp(-e^{t})$, $t \in \mathbb{R}$

b) Let now F be a generic cdf. Which of the following functions are guaranteed to be also cdfs?

5. $F_5(t) = F(t^2)$, $t \in \mathbb{R}$
6. $F_6(t) = F(t)^2$, $t \in \mathbb{R}$
7. $F_7(t) = F(1 - \exp(-t))$, $t \in \mathbb{R}$
8. $F_8(t) = \begin{cases} 1 - \exp(-F(t)/(1-F(t))) & \text{if } F(t) < 1 \\ 1 & \text{if } F(t) = 1 \end{cases} \quad t \in \mathbb{R}$

Exercise 2. Let $X_1, \ldots, X_n$ be i.i.d. $\sim \mathcal{E}(1)$ random variables (i.e., they are independent and identically distributed, all with the exponential distribution of parameter $\lambda = 1$).

a) Compute the cdf of $Y_n = \min\{X_1, \ldots, X_n\}$.

b) How do $\mathbb{P}(\{Y_n \leq t\})$ and $\mathbb{P}(\{X_1 \leq t\})$ compare when n is large and t is such that $t \ll \frac{1}{n}$?

c) Compute the cdf of $Z_n = \max\{X_1, \ldots, X_n\}$.

d) How do $\mathbb{P}(\{Z_n \geq t\})$ and $\mathbb{P}(\{X_1 \geq t\})$ compare when n is large and t is such that $t \gg \log(n)$?

Exercise 3.*

a) Let X, Y be two random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G} = \sigma(X) \cap \sigma(Y)$ [fact: it can be shown that $\mathcal{G}$ is a σ-field]. Is it true that $\{X \leq Y\} \in \mathcal{G}$?

b) Let X, Y be two independent random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Is it always true that $\sigma(X + Y) = \sigma(X, Y)$?

c) Let X be a continuous random variable whose pdf p_X is a continuous function on $\mathbb{R}$. Let now $Y = X^2$. Is it always true that the pdf p_Y is also a continuous function on $\mathbb{R}$?

d) Let F be a generic cdf. Is it always true that the function $G : \mathbb{R} \to [0, 1]$ defined as

$$G(t) = F(t^3 + 3t^2 + 3t + 1), \quad t \in \mathbb{R}$$

is also a cdf ?

Exercise 4. Let $n \geq 1$, $\Omega = \{1, 2, \dots, n\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ and $\mathbb{P}$ be the probability measure on $(\Omega, \mathcal{F})$ defined by $\mathbb{P}(\{\omega\}) = \frac{1}{n}$ on the singletons and extended by additivity to all subsets of Ω.

a) Consider first $n = 4$. Find three subsets $A_1, A_2, A_3 \subset \Omega$ such that

$$\mathbb{P}(A_j \cap A_k) = \mathbb{P}(A_j) \cdot \mathbb{P}(A_k) \quad \forall j \neq k \quad \text{but} \quad \mathbb{P}(A_1 \cap A_2 \cap A_3) \neq \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \mathbb{P}(A_3)$$

b) Consider now $n = 6$. Find three subsets $A_1, A_2, A_3 \subset \Omega$ such that

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \mathbb{P}(A_3) \quad \text{but} \quad \exists j \neq k \text{ such that } \mathbb{P}(A_j \cap A_k) \neq \mathbb{P}(A_j) \cdot \mathbb{P}(A_k)$$

c) Consider finally a generic probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and three events $A_1, A_2, A_3 \in \mathcal{F}$ such that

$$\mathbb{P}(A_j \cap A_k) = \mathbb{P}(A_j) \cdot \mathbb{P}(A_k) \quad \forall j \neq k \quad \text{and} \quad \mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \mathbb{P}(A_3)$$

Show that A_1, A_2, A_3 are independent according to the definition given in the course.

Exercise 5. Let X_1, X_2 be two i.i.d. random variables such that $\mathbb{P}(\{X_i = +1\}) = \mathbb{P}(\{X_i = -1\}) = 1/2$ for $i = 1, 2$. Let also $Y = X_1 + X_2$ and $Z = X_1 - X_2$.

a) Are Y and Z independent?

b) Same question with X_1, X_2 i.i.d.$\sim \mathcal{N}(0, 1)$ random variables (use here the change of variable formula in order to compute the joint distribution of Y and Z).