Solutions to Homework 2

Exercise 1. a) 1. true, 2. false, 3. false, 4. true b) 5. false, 6. true, 7. false, 8. true.

Exercise 2. a) We have

$$\begin{aligned}\mathbb{P}(\{Y_n \le t\}) &= 1 - \mathbb{P}(\{Y_n > t\}) = 1 - \mathbb{P}(\{\min\{X_1, \ldots, X_n\} > t\}) = 1 - \mathbb{P}(\cap_{j=1}^n \{X_j > t\}) \\ &= 1 - \textstyle\prod_{j=1}^n \mathbb{P}(\{X_j > t\}) = 1 - \mathbb{P}(\{X_1 > t\})^n\end{aligned}$$

where the last two equalities follow from the assumption that the X's are i.i.d. Therefore,

$$\mathbb{P}(\{Y_n \le t\}) = 1 - (\exp(-t))^n = 1 - \exp(-nt)$$

b) Under the assumptions made, n is large and t is such that $nt \ll 1$, so using Taylor's expansion $\exp(-x) \simeq 1 - x$, we obtain

$$\mathbb{P}(\{Y_n \le t\}) \simeq 1 - (1 - nt) = nt \quad \text{while} \quad \mathbb{P}(\{X_1 \le t\}) = 1 - \exp(-t) \simeq t$$

and therefore $\mathbb{P}(\{Y_n \le t\}) \simeq n\,\mathbb{P}(\{X_1 \le t\})$.

c) We have similarly

$$\begin{aligned}\mathbb{P}(\{Z_n \ge t\}) &= 1 - \mathbb{P}(\{Z_n < t\}) = 1 - \mathbb{P}(\{\max\{X_1, \ldots, X_n\} < t\}) = 1 - \mathbb{P}(\cap_{j=1}^n \{X_j < t\}) \\ &= 1 - \prod_{j=1}^n \mathbb{P}(\{X_j < t\}) = 1 - \mathbb{P}(\{X_1 < t\})^n = 1 - (1 - \exp(-t))^n\end{aligned}$$

d) Under the assumptions made, n is large and t is such that $n\exp(-t) \ll 1$, so using again the same Taylor expansion as above, we obtain

$$\mathbb{P}(\{Z_n \ge t\}) \simeq 1 - (1 - n\,\exp(-t)) = n\,\exp(-t) \quad \text{while} \quad \mathbb{P}(\{X_1 \ge t\}) = \exp(-t)$$

and therefore $\mathbb{P}(\{Z_n \ge t\}) \simeq n\,\mathbb{P}(\{X_1 \ge t\})$.

Exercise 3.

a) No. Take for example $\Omega = \{1, 2, 3\}$, $X(\omega) = \omega$ and $Y(\omega) = 2$ for every $\omega \in \Omega$. Then $\mathcal{G} = \sigma(X) \cap \sigma(Y) = \sigma(Y) = \{\emptyset, \Omega\}$, but $\{X \le Y\} = \{\omega \in \Omega : X(\omega) \le Y(\omega)\} = \{1, 2\} \notin \mathcal{G}$.

b) No. Take for example $\Omega = \{1, 2\}^2$, $X(\omega) = \omega_1$ and $Y(\omega) = -\omega_2$. Then $\{X + Y = 0\} = \{(1, 1), (2, 2)\}$, and so $\sigma(X + Y) = \sigma(\{(1, 1), (2, 2)\}, \{(1, 2)\}, \{(2, 1)\}) \ne \sigma(X, Y) = \mathcal{P}(\Omega)$ (in addition, note that the fact that X and Y are independent does not play a role here).

c) No. Take for example $X \sim \mathcal{N}(0, 1)$, whose pdf $p_X(x) = \frac{1}{\sqrt{2\pi x}} \exp(-x^2/2)$ is continuous. Then $Y = X^2$ has pdf

$$p_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi y}} \exp(-y/2) & \text{if } y \ge 0 \\ 0 & \text{if } y < 0 \end{cases}$$

which is discontinuous in $y = 0$.

d) Yes. Actually, the map $t \mapsto t^3 + 3t^2 + 3t + 1 = (t+1)^3$ is non-decreasing and going from $-\infty$ to $+\infty$, thus the properties of the cdf F are preserved for G.

Exercise 4. a) Here are 3 possible subsets A_1, A_2, A_3 of $\Omega = \{1,2,3,4\}$: $A_1 = \{1,2\}$, $A_2 = \{1,3\}$ and $A_3 = \{1,4\}$. We check that

$$\mathbb{P}(A_j) = \frac{1}{2} \quad \forall j \quad \text{and} \quad \mathbb{P}(A_j \cap A_k) = \frac{1}{4} = \mathbb{P}(A_j) \cdot \mathbb{P}(A_k) \quad \forall j \neq k$$

but

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \frac{1}{4} \neq \frac{1}{8} = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \mathbb{P}(A_3)$$

b) Here are 3 possible subsets A_1, A_2, A_3 of $\Omega = \{1,2,3,4,5,6\}$: $A_1 = \{1,2,3\}$, $A_2 = \{3,4,5\}$ and $A_3 = \{1,3,4,6\}$. We check that

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \frac{1}{6} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{2}{3} = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \mathbb{P}(A_3)$$

but

$$\mathbb{P}(A_1 \cap A_2) = \frac{1}{6} \neq \frac{1}{4} = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2)$$

c) Using the assumptions made, we check successively (the roles of A_1, A_2, A_3 being permutable):

$$\begin{aligned}
\mathbb{P}(A_1 \cap A_2 \cap A_3^c) &= \mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) - \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \mathbb{P}(A_3) \\
&= \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot (1 - \mathbb{P}(A_3)) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \mathbb{P}(A_3^c) \\
\mathbb{P}(A_1 \cap A_2^c \cap A_3^c) &= \mathbb{P}(A_1 \cap A_3^c) - \mathbb{P}(A_1 \cap A_2 \cap A_3^c) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_3^c) - \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \mathbb{P}(A_3^c) \\
&= \mathbb{P}(A_1) \cdot (1 - \mathbb{P}(A_2)) \cdot \mathbb{P}(A_3^c) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2^c) \cdot \mathbb{P}(A_3^c) \\
\mathbb{P}(A_1^c \cap A_2^c \cap A_3^c) &= \mathbb{P}(A_2^c \cap A_3^c) - \mathbb{P}(A_1 \cap A_2^c \cap A_3^c) = \mathbb{P}(A_2^c) \cdot \mathbb{P}(A_3^c) - \mathbb{P}(A_1) \cdot \mathbb{P}(A_2^c) \cdot \mathbb{P}(A_3^c) \\
&= (1 - \mathbb{P}(A_1)) \cdot \mathbb{P}(A_2^c) \cdot \mathbb{P}(A_3^c) = \mathbb{P}(A_1^c) \cdot \mathbb{P}(A_2^c) \cdot \mathbb{P}(A_3^c)
\end{aligned}$$

Exercise 5. a) No. Even though it is easily shown that Y and Z are uncorrelated random variables (i.e., that their covariance is zero), they are not independent. Here is a counter-example: $\mathbb{P}(\{Y = +2\}) = \mathbb{P}(\{Z = +2\}) = 1/4$, but $\mathbb{P}(\{Y = +2,\ Z = +2\}) = 0$. So we have found two Borel sets $B_1 = \{+2\}$ and $B_2 = \{+2\}$ such that

$$\mathbb{P}(\{Y \in B_1,\ Z \in B_2\}) \neq \mathbb{P}(\{Y \in B_1\}) \cdot \mathbb{P}(\{Z \in B_2\})$$

b) Yes. In this case again, one checks easily that Y and Z are uncorrelated. Let us now compute their joint pdf: the joint pdf of X_1 and X_2 is given by

$$p_{X_1,X_2}(x_1, x_2) = \frac{1}{2\pi} \exp\left(-\frac{x_1^2 + x_2^2}{2}\right)$$

Making now the change of variables $y = x_1 + x_2$, $z = x_1 - x_2$, or equivalently $x_1 = \frac{y+z}{2}$, $x_2 = \frac{y-z}{2}$, we obtain

$$x_1^2 + x_2^2 = \left(\frac{y+z}{2}\right)^2 + \left(\frac{y-z}{2}\right)^2 = \frac{y^2 + z^2}{2}$$

and the Jacobian of this linear transformation is given by

$$J(y,z) = \det \begin{pmatrix} \frac{\partial x_1}{\partial y} & \frac{\partial x_2}{\partial y} \\ \frac{\partial x_1}{\partial z} & \frac{\partial x_2}{\partial z} \end{pmatrix} = \det \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} = -\frac{1}{2}$$

so that

$$p_{Y,Z}(y,z) = p_{X_1,X_2}(x_1(y,z), x_2(y,z)) \cdot |J(y,z)| = \frac{1}{4\pi} \exp\left(-\frac{y^2+z^2}{4}\right)$$

from which we deduce that Y and Z are *independent* $\mathcal{N}(0,2)$ random variables.