

Homework 3

Exercise 1. Let $\Omega = \mathbb{R}^2$ and $\mathcal{F} = \mathcal{B}(\mathbb{R}^2)$. Let also $X_1(\omega) = \omega_1$ and $X_2(\omega) = \omega_2$ for $\omega = (\omega_1, \omega_2) \in \Omega$ and let finally μ be a probability distribution on \mathbb{R} . We consider below two different probability measures defined on (Ω, \mathcal{F}) , defined on the “rectangles” $B_1 \times B_2$ (Caratheodory’s extension theorem then guarantees that these probability measures can be extended uniquely to $\mathcal{B}(\mathbb{R}^2)$).

a) $\mathbb{P}^{(1)}(B_1 \times B_2) = \mu(B_1) \cdot \mu(B_2)$

b) $\mathbb{P}^{(2)}(B_1 \times B_2) = \mu(B_1 \cap B_2)$

In each case, describe what is the relation between the random variables X_1 and X_2 .

Exercise 2. Let X_1, X_2 be two independent and identically distributed (i.i.d.) $\mathcal{N}(0, 1)$ random variables. Compute the pdf of $X_1 + X_2$ (using convolution).

Exercise 3. Let X, Y be two i.i.d. $\mathcal{N}(0, 1)$ random variables, and Z be independent of X, Y and such that $\mathbb{P}\{Z = +1\} = \mathbb{P}\{Z = -1\} = 1/2$.

a) Is it true $X + ZY$ is a Gaussian random variable ?

b) Is it true that $X + ZY$ and Y are independent random variables ?

Exercise 4.* Let $\Omega = \{1, \dots, n\}$, $\mathcal{F} = \mathcal{P}(\Omega)$, and $\mathbb{P}(\{\omega\}) = \frac{1}{n}$ for all $\omega \in \Omega$. Let $X(\omega) = \omega$.

a) Let $n = 6$, and define

$$Y_2 = X \pmod{2}, \quad \text{and} \quad Y_3 = X \pmod{3}.$$

Are Y_2 and Y_3 independent?

b) Describe $\sigma(\{Y_2\})$ and $\sigma(\{Y_3\})$.

c) Let $n = 30$, and define

$$Y_2 = X \pmod{2}, \quad Y_3 = X \pmod{3}, \quad \text{and} \quad Y_5 = X \pmod{5}.$$

Are Y_i and Y_j pairwise independent for $i \neq j$, $i, j \in \{2, 3, 5\}$? Are Y_2, Y_3, Y_5 (jointly) independent?