Solutions to Homework 3

Exercise 1. a) We have

$$\mathbb{E}(Y) = \mathbb{E}(X^a) = \int_0^{+\infty} x^a \, \lambda \, \exp(-\lambda x) \, dx < +\infty \quad \text{if and only if} \quad a > -1$$

b) Likewise:

$$\mathbb{E}(Y^2) = \mathbb{E}(X^{2a}) = \int_0^{+\infty} x^{2a} \, \lambda \, \exp(-\lambda x) \, dx < +\infty \quad \text{if and only if} \quad a > -\frac{1}{2}$$

c) Therefore, c1) $\text{Var}(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2$ is well defined and finite $\forall a > -\frac{1}{2}$; c2) $\text{Var}(Y)$ is well defined but takes the value $+\infty$ for $-\frac{1}{2} \geq a > -1$, and c3) $\text{Var}(Y)$ is ill-defined (indetermination of the type $\infty - \infty$) for $a \leq -1$.

d) The only integer values of a for which $\mathbb{E}(Y)$ and $\text{Var}(Y)$ are well-defined are non-negative values. For $a = 0$, we have $Y = X^0 = 1$, so $\mathbb{E}(Y) = 1$ and $\text{Var}(Y) = 0$. For $a \geq 1$, we obtain by integration by parts:

$$\mathbb{E}(Y) = \mathbb{E}(X^a) = \int_0^{+\infty} x^a \, \lambda \, \exp(-\lambda x) \, dx$$

$$= \int_0^{+\infty} \frac{a}{\lambda} \, x^{a-1} \, \lambda \, \exp(-\lambda x) \, dx = \ldots = \frac{a!}{\lambda^a} \cdot 1$$

so

$$\mathbb{E}(Y^2) = \mathbb{E}(X^{2a}) = \frac{(2a)!}{\lambda^{2a}} \quad \text{and} \quad \text{Var}(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 = \frac{(2a)! - (a!)^2}{\lambda^{2a}}$$

Exercise 2. First note that as $X \sim -X$, it holds that $\mathbb{P}(\{X \geq 0\}) \geq \frac{1}{2}$ and $\mathbb{E}(X) = 0$.

a) $\text{Cov}(X, Y) = \mathbb{E}(X \, 1_{\{X \geq 0\}}) \geq 0$ as $X \, 1_{\{X \geq 0\}}$ is a non-negative random variable.

b) Using the suggested inequality, we find

$$\text{Cov}(X, Y) \leq \sqrt{\text{Var}(X)} \, \sqrt{\text{Var}(Y)} = \sqrt{1} \, \sqrt{\mathbb{P}(\{X \geq 0\}) - \mathbb{P}(\{(X \geq 0\})^2} \leq \sqrt{\frac{1}{4}} = \frac{1}{2} = C$$

as $\mathbb{P}(\{X \geq 0\}) - \mathbb{P}(\{(X \geq 0\})^2 \leq \frac{1}{4}$ (which is maximized when $\mathbb{P}(\{X \geq 0\}) = \frac{1}{2}$).

c) The computation gives

$$\text{Cov}(X, Y) = \mathbb{E}(X \, 1_{\{X \geq 0\}}) = \int_0^{+\infty} x \, \frac{1}{\sqrt{2\pi}} \, \exp(-x^2/2) \, dx = \frac{1}{\sqrt{2\pi}} \left(- \exp(-x^2/2)\right)\Big|_{x=0}^{x=+\infty} = \frac{1}{\sqrt{2\pi}}$$

(clearly satisfying the above two inequalities)

d) The answer to the first question is yes: take X such that $\mathbb{P}(\{X = +1\}) = \mathbb{P}(\{X = -1\}) = \frac{1}{2}$ (verifying $X \sim -X$, $\text{Var}(X) = 1$ and $\text{Cov}(X, Y) = \frac{1}{2}$).

e) The answer to the first question is no, but the one to the second is yes: consider X_n such that $\mathbb{P}(\{X_n = n\}) = \mathbb{P}(\{X_n = -n\}) = \frac{1}{2n^2}$ and $\mathbb{P}(\{X_n = 0\}) = 1 - \frac{1}{n^2}$. Then $X_n \sim -X_n$ and

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$\text{Var}(X_n) = 1$ for every n, and $\text{Cov}(X_n, Y_n) = \mathbb{E}(X_n 1_{\{X_n \geq 0\}}) = n \frac{1}{2n^2} = \frac{1}{2n} \xrightarrow[n \to \infty]{} 0$.

Exercise 3. a) Using the formula given in the problem set, we obtain:

$$\mathbb{E}(X) = \int_0^{+\infty} \exp(-\lambda t)\, dt = \frac{1}{\lambda}$$

b) Using the formula given in the problem set together with the fact that X is integer-valued, we obtain:

$$\mathbb{E}(X) = \sum_{k \geq 0} \int_k^{k+1} \mathbb{P}(\{X \geq t\})\, dt = \sum_{k \geq 0} \int_k^{k+1} \mathbb{P}(\{X \geq k+1\})\, dt = \sum_{k \geq 0} \mathbb{P}(\{X \geq k+1\}) = \sum_{k \geq 1} \mathbb{P}(\{X \geq k\})$$

c) Applying the above formula, we obtain in the first case ($X \sim \text{Bern}(p)$):

$$\mathbb{E}(X) = \mathbb{P}(\{X \geq 1\}) = p$$

In the second case ($X \sim \text{Geom}(p)$), we obtain:

$$\mathbb{E}(X) = \sum_{k \geq 1} \sum_{m \geq k} p^m (1-p) = \sum_{k \geq 1} p^k \sum_{m \geq k} p^{m-k} (1-p) = \sum_{k \geq 1} p^k \frac{1}{1-p} 1 - p = \frac{1}{1-p} - 1 = \frac{p}{1-p}$$

Exercise 4. a) The computation of the characteristic function gives in this case:

$$\phi_X(t) = \sum_{k \geq 0} \frac{\lambda^k e^{-\lambda}}{k!} e^{itk} = \sum_{k \geq 0} \frac{(\lambda e^{it})^k e^{-\lambda}}{k!} = e^{\lambda e^{it}} e^{-\lambda} = e^{\lambda(e^{it}-1)}$$

b) The general expression for ϕ_X is given by

$$\phi_X(t) = \sum_{\ell \in \mathbb{Z}} \mathbb{P}(\{X = \ell\}) e^{it\ell}$$

Plugging this expression into the proposed formula, we find

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-itk} \phi_X(t)\, dt = \sum_{\ell \in \mathbb{Z}} \mathbb{P}(\{X = \ell\}) \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-it(\ell-k)}\, dt = \sum_{l \in \mathbb{Z}} \mathbb{P}(\{X = \ell\}) \delta_{k\ell} = \mathbb{P}(\{X = k\})$$

where we have switched the sum and integral without too much checking and we have used the fact that for $k \neq \ell$:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-it(\ell-k)}\, dt = \frac{1}{2\pi} \frac{e^{ik(\ell-k)}}{i(\ell-k)} \Big|_{t=-\pi}^{t=\pi} = 0$$

c) Let us compute

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-itk} \cos(t)\, dt = \frac{1}{4\pi} \int_{-\pi}^{\pi} e^{-itk} (e^{it} + e^{-it})\, dt$$

$$= \frac{1}{4\pi} \int_{-\pi}^{\pi} (e^{-it(k-1)} + e^{-it(k+1)})\, dt = \begin{cases} \frac{1}{2} & \text{if } k \in \{-1, +1\} \\ 0 & \text{otherwise} \end{cases}$$

by the same argument as above.

d) We know that $\phi_X(t) = \cos(t)$ is a characteristic function because $\phi_X(0) = \cos(0) = 1$, ϕ_X is continuous on $\mathbb{R}$, and also positive semi-definite. Indeed, using the trigonometric identity $\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b)$, we obtain

$$\sum_{j,k=1}^{n} c_j \overline{c_k} \, \phi_X(t_j - t_k) = \sum_{j,k=1}^{n} c_j \overline{c_k} \cos(t_j - t_k) = \sum_{j,k=1}^{n} c_j \overline{c_k} \left(\cos(t_j) \cos(t_k) + \sin(t_j) \sin(t_k) \right)$$

$$= \left| \sum_{j=1}^{n} c_j \cos(t_j) \right|^2 + \left| \sum_{j=1}^{n} c_j \sin(t_j) \right|^2 \geq 0$$

for every $n \geq 1$, $t_1, \ldots, t_n \in \mathbb{R}$ and $c_1, \ldots, c_n \in \mathbb{C}$.

Exercise 5. a) i) From the course, we know that if $\mathbb{E}(|X|) < +\infty$, then ϕ_X is continuously differentiable on $\mathbb{R}$. Using the contraposition, we deduce that $\mathbb{E}(|X|) = +\infty$ here.

a) ii) From the course again, the fact that ϕ_X is integrable on $\mathbb{R}$ implies that X admits a pdf p_X.

b) By the inversion formula seen in class, we have

$$p_X(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} e^{-\lambda|t|} \, dt = \frac{1}{2\pi} \left(\int_{-\infty}^{0} e^{-t(ix-\lambda)} \, dt + \int_{0}^{+\infty} e^{-t(ix+\lambda)} \, dt \right)$$

$$= \frac{1}{2\pi} \left(-\frac{1}{ix - \lambda} + \frac{1}{ix + \lambda} \right) = \frac{1}{\pi} \frac{\lambda}{x^2 + \lambda^2}$$

This pdf is the that of a (centered) Cauchy distribution with parameter λ (also known as Lorentz distribution in physics). The word "centered" is a bit misleading here, as we have seen in part a)i) that $E(|X|) = +\infty$ (which can also been checked directly from the expression of p_X), so that $\mathbb{E}(X)$ is ill-defined. Nevertheless, the pdf appears to have a peak clearly centered in $x = 0$ here, and writing $\mathbb{E}(X) = 0$ can actually be justified via a more general definition of expectation. Besides, the parameter $\lambda > 0$ is connected to the width of the peak, but is by no means connected to the standard deviation of the random variable X, which is truly infinite.

c) Using the change of variable formula, we obtain

$$p_Y(x) = p_{1/X}(x) = p_X\left(\frac{1}{x}\right) \cdot \left| -\frac{1}{x^2} \right| = \frac{1}{\pi} \frac{\lambda}{x^{-2} + \lambda^2} \frac{1}{x^2}$$

$$= \frac{1}{\pi} \frac{\lambda}{1 + \lambda^2 x^2} = \frac{1}{\pi} \frac{\lambda^{-1}}{\lambda^{-2} + x^2}$$

so we see that Y is again a Cauchy random variable, with parameter $1/\lambda$.

d) By the factorization property of characteristic functions, we obtain

$$\phi_{X_1 + \ldots + X_n}(t) = \prod_{i=1}^{n} \phi_{X_i}(t) = (\phi_X(t))^n = \exp(-\lambda n|t|)$$

so $X_1 + \ldots + X_n$ is also a Cauchy random variable with parameter λn, and $Z_n = \frac{X_1 + \ldots + X_n}{n}$ is a Cauchy random variable with parameter λ, for every $n \geq 1$. Similarly, we obtain, using part b),

$$\phi_{1/X_1 + \ldots + 1/X_n}(t) = (\phi_{1/X}(t))^n = \exp(-n|t|/\lambda)$$

so $1/X_1 + \ldots + 1/X_n$ is a Cauchy random variable with parameter n/λ. Therefore, again by part b), $\frac{1}{1/X_1 + \ldots + 1/X_n}$ is a Cauchy random variable with parameter λ/n and $W_n = \frac{n}{1/X_1 + \ldots + 1/X_n}$ is (again) a Cauchy random variable with parameter λ.

e) The first oddity of the above results is that the empirical average $Z_n = \frac{X_1 + \ldots + X_n}{n}$ does not converge to a limit as n goes to infinity. One reason for this is that $\mathbb{E}(|X|) = +\infty$, so the law of large numbers does not hold, as we shall see later in the course. The second oddity is that the sum of an arbitrary number of Cauchy random variables is still a Cauchy random variable. The other well known distribution sharing this property is the Gaussian distribution, but that's basically it, as this property is an exception among probability distributions. The third oddity is that the *arithmetic mean* Z_n of the random variables $X_1, \ldots, X_n$ has the same distribution as their *harmonic mean* W_n. However, as we deal here with random variables taking positive and negative values, the classical inequality "arithmetic mean $\geq$ harmonic mean" does not hold, so there is no contradiction.