Solutions to Homework 5

Exercise 1. a). **True**; Note that for any random variable X, and a function of X, say $f(X)$ we have $\sigma(X, f(X)) = \sigma(X)$. Thus, $\sigma(X_1, X_2) = \sigma(X_1, X_2, R, S, T)$ as R, S, T are functions of X_1, X_2.

b). **False**; For any random variable X and a surjective function of X, say $f(X)$ we have that $\sigma(f(X)) \subsetneq \sigma(X)$. Here, the absolute value function, $|\cdot|$, is not injective thus information about the sign of the random variables X_1, X_2 is lost in Y_1, Y_2. Thus, $\sigma(Y_1, Y_2) \subsetneq \sigma(X_1, X_2)$.

c). **True**; Note that $R = X_1 + X_2$ and $S = X_1 - X_2$. Thus, $\sigma(R, S) = \sigma(R + S, R - S)$ as the mapping from $(R, S) \to (R + S, R - S)$ is a bijection. Further, $\sigma(R + S, R - S) = \sigma(X, Y)$.

d). **False**; Consider the case where $R = 0$ which implies $T \leq 0$. Thus $X_1 = -X_2$, and having the additional information about the absolute values of X_1 and X_2 (i.e., Y_1 and Y_2, respectively) doesn't provide any information about the signs of the random variables X_1 and X_2.

e). **False**; We will follow the same approach as in the part (d) to see if we can reconstruct information about X_1 and X_2 from Y_1, Y_2, S and T. Consider the case where $S = 0$ which also implies $T \geq 0$. Thus, $X_1 = X_2$. However, even after having the additional information about Y_1 and Y_2, it doesn't tells us whether the values taken by X_1 and X_2 are positive or negative.

Exercise 2. a) Option 1: by the assumptions made, $\text{Cov}(X_1 + X_2, X_1 - X_2) = \text{Var}(X_1) + \text{Cov}(X_2, X_1) - \text{Cov}(X_1, X_2) - \text{Var}(X_2) = \text{Var}(X_1) - \text{Var}(X_2) = 0$. Besides, as X_1, X_2 are independent Gaussian random variables, $X = (X_1, X_2)$ is a Gaussian random vector, so $(X_1 + X_2, X_1 - X_2)$ is also a Gaussian random vector whose components are uncorrelated, and therefore independent, by Proposition 6.8 of the course.

Option 2 is to show directly that

$$\mathbb{E}(e^{it_1(X_1+X_2)+it_2(X_1-X_2)}) = \mathbb{E}(e^{it_1(X_1+X_2)})\,\mathbb{E}(e^{it_2(X_1-X_2)}) \quad \forall t_1, t_2 \in \mathbb{R}$$

as this would imply independence of $X_1 + X_2$ and $X_1 - X_2$. We check indeed that

$$\begin{aligned}
&\mathbb{E}(e^{it_1(X_1+X_2)+it_2(X_1-X_2)}) = \mathbb{E}(e^{i(t_1+t_2)X_1+i(t_1-t_2)X_2}) \\
&= \mathbb{E}(e^{i(t_1+t_2)X_1})\,\mathbb{E}(e^{i(t_1-t_2)X_2}) = e^{i\mu_1(t_1+t_2)-\sigma_1^2(t_1+t_2)^2/2}\; e^{i\mu_2(t_1-t_2)-\sigma_2^2(t_1-t_2)^2/2}
\end{aligned}$$

Because of the assumption made ($\sigma_1^2 = \sigma_2^2 = \sigma^2$), the above expression is further equal to

$$\begin{aligned}
&= e^{i(\mu_1+\mu_2)t_1+i(\mu_1-\mu_2)t_2-\sigma^2(t_1^2+t_2^2)} = e^{i(\mu_1+\mu_2)t_1-\sigma^2 t_1^2}\; e^{i(\mu_1-\mu_2)t_2-\sigma^2 t_2^2} \\
&= \mathbb{E}(e^{it_1(X_1+X_2)})\,\mathbb{E}(e^{it_2(X_1-X_2)})
\end{aligned}$$

which proves the claim.

b) 1. Skipped. Just note that closing our eyes, we could compute

$$\phi_X'(t) = i\,\mathbb{E}(X\,e^{itX}) \quad \text{and} \quad \phi_X''(t) = -\mathbb{E}(X^2 e^{itX}), \quad t \in \mathbb{R}$$

and deduce from there that indeed, if $\mathbb{E}(X^2) < +\infty$, then ϕ_X is twice continuously differentiable. As a by-product, we obtain the relation

$$\phi_X''(0) = -\mathbb{E}(X^2)$$

from the second formula evaluated in $t = 0$.

2. Skipped.

3. By the assumptions made, we obtain

$$\mathbb{E}(e^{it_1(X_1+X_2)+it_2(X_1-X_2)}) = \mathbb{E}(e^{it_1(X_1+X_2)})\,\mathbb{E}(e^{it_2(X_1-X_2)})$$

and also

$$\mathbb{E}(e^{it_1(X_1+X_2)+it_2(X_1-X_2)}) = \mathbb{E}(e^{i(t_1+t_2)X_1+i(t_1-t_2)X_2}) = \phi_{X_1}(t_1+t_2)\,\phi_{X_2}(t_1-t_2)$$

so

$$\log\phi_{X_1}(t_1+t_2) + \log\phi_{X_2}(t_1-t_2) = \log\mathbb{E}(e^{it_1(X_1+X_2)}) + \log\mathbb{E}(e^{it_2(X_1-X_2)}) = g_1(t_1) + g_2(t_2)$$

proving the claim.

4. Differentiating first the equality with respect to t_1, we obtain

$$f_1'(t_1+t_2) + f_2'(t_1-t_2) = g_1'(t_1)$$

and then with respect to t_2:

$$f_1''(t_1+t_2) - f_2''(t_1-t_2) = 0$$

Setting $t_1 = t_2 = \frac{t}{2}$ leads to $f_1''(t) = f_2''(0)$, and setting $t_1 = -t_2 = \frac{t}{2}$ leads to $f_2''(t) = f_1''(0)$. As these equalities are satisfied for arbitrary $t \in \mathbb{R}$, this says that the second derivatives of both f_1 and f_2 are constant functions, therefore that both f_1 and f_2 are polynomials of degree less than or equal to 2.

5. The assumption is that $\log\phi_X(t) = at^2 + bt + c$ for $t \in \mathbb{R}$. Using the hint and writing $\mu = \mathbb{E}(X)$, $\sigma^2 = \mathrm{Var}(X)$, we obtain successively:

$$\begin{aligned} e^c = \phi_X(0) &= 1 & \text{so } c = 0 \\ b = \phi_X'(0) &= i\mu & \text{so } b = i\mu \\ 2a + b^2 = \phi_X''(0) &= -\mathbb{E}(X^2) = -(\mu^2+\sigma^2) & \text{so } a = -\sigma^2/2 \end{aligned}$$

Therefore, $\phi_X(t) = e^{i\mu t - \sigma^2 t^2/2}$, which is the characteristic function of a Gaussian.

6. As X_1, X_2 are independent and Gaussian, this implies that (X_1, X_2) is a Gaussian vector, i.e., that X_1, X_2 are jointly Gaussian. By the assumptions made, we also have

$$0 = \mathrm{Cov}(X_1+X_2, X_1-X_2) = \mathrm{Var}(X_1)+\mathrm{Cov}(X_2,X_1)-\mathrm{Cov}(X_1,X_2)-\mathrm{Var}(X_2) = \mathrm{Var}(X_1)-\mathrm{Var}(X_2)$$

so $\mathrm{Var}(X_1) = \mathrm{Var}(X_2)$ [note in passing that we did not use here the assumption that X_1 and X_2 are uncorrelated]. This finally completes the proof of the result stated in part b).

Exercise 3. a) Using $\psi(x) = x^2$ or $\psi(x) = \sigma^2 + x^2$ in Chebyshev's inequality leads to respectively

$$\mathbb{P}(\{X \ge t\}) \le \frac{\sigma^2}{t^2} \quad \text{and} \quad \mathbb{P}(\{X \ge t\}) \le \frac{2\sigma^2}{\sigma^2 + t^2}$$

which is not what we want. Using the hint (with $b \ge 0$ in order to satisfy the hypotheses), we obtain

$$\mathbb{P}(\{X \ge t\}) \le \frac{\mathbb{E}((X+b)^2)}{(t+b)^2} = \frac{\sigma^2 + b^2}{(t+b)^2}$$

Optimizing over the parameter b, we find that best possible bound is obtained by setting $b^* = \frac{\sigma^2}{t}$ (which is non-negative), leading to

$$\mathbb{P}(\{X \geq t\}) \leq \frac{\sigma^2}{\sigma^2 + t^2}$$

b) Using Cauchy-Schwarz's inequality with the random variables X and $Y = 1_{\{X>t\}}$, we obtain

$$\mathbb{E}(X\,1_{\{X>t\}})^2 \leq \mathbb{E}(X^2)\,\mathbb{P}(\{X > t\})$$

On the other hand, we have $\mathbb{E}(X\,1_{\{X>t\}}) = \mathbb{E}(X) - \mathbb{E}(X\,1_{\{X\leq t\}}) \geq \mathbb{E}(X) - t$, therefore the result.