Exercise Set 2 : Solution
Quantum Computation

Exercise 1 *Matrix representation of a few gates / circuits*

(a) The Hilbert space here is $\mathbb{C}^8$ and its matrix representation in the computational basis $\left\{|000\rangle, |001\rangle, |010\rangle, |011\rangle, |010\rangle, |101\rangle, |110\rangle, |111\rangle\right\}$ is given by

$$CCNOT = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(b) The matrix representation of this circuit in the computational basis $\left\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\right\}$ of $\mathbb{C}^4$ is given by

$$NOT_y = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(c) The matrix representation of this circuit (in the same basis) is given by

$$NOT_x = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

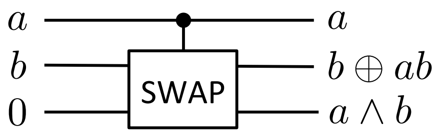
(d) The matrix representation of this circuit (in the same basis) is given by

$$NOT_x \cdot CNOT \cdot NOT_x = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

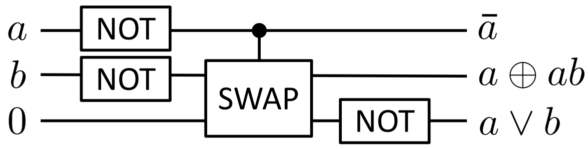
All the above matrices are permutation matrices, and are also equal to their own inverse.

Exercise 2 *Fredkin gate*

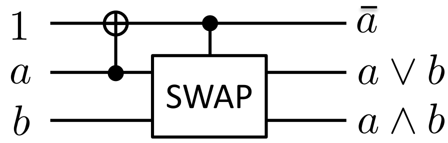
(a) The AND gate can be represented as follows with only the Fredkin gate :



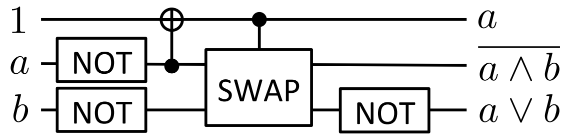
The OR gate is then (using $a \vee b = \mathrm{NOT}\big(\mathrm{NOT}(a) \wedge \mathrm{NOT}(b)\big)$)



Another solution for both AND and OR uses a combination of CSWAP and CNOT :



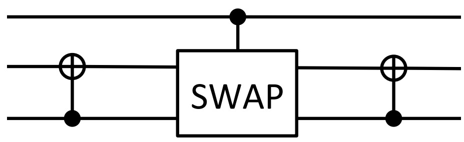
For the OR gate, alternatively, we then have :



(b) The Fredkin is a controlled SWAP which swap's the last two bits if the first one is equal to 1. Thus we find

$$\mathrm{CSWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(c) From the matrix representation of Fredkin, we see that to obtain the matrix representation of CCNOT, we have to permute on rows 5,6,7,8. With a bit of thought one can find that the CCNOT gate can be represented as



Another way is by noting that

$$\begin{aligned}\text{CNOT}|x,y\rangle &= |x, x\oplus y\rangle,\\ \text{CCNOT}|x,y,z\rangle &= |x,y,z\oplus xy\rangle,\\ \text{CSWAP}|x,y,z\rangle &= |x, y\oplus x(y\oplus z), z\oplus x(y\oplus z)\rangle.\end{aligned}$$

Thus an input $|x,y,z\rangle$ becomes $|x,y\oplus z,z\rangle$ after the first CNOT gate, $|x,y\oplus z\oplus xy, z\oplus xy\rangle$ after the Fredkin gate and $|x,y,z\oplus xy\rangle$ after the second CNOT gate.

Exercise 3 *Mach-Zehnder interferometer*

(a) A matrix U is unitary if $UU^\dagger = U^\dagger U = I$. Note that for Hadamard and NOT(X) gates, we have $HH^\dagger = H^\dagger H = I$, $XX^\dagger = X^\dagger X = I$. For HXH, we have

$$HXH(HXH)^\dagger = HXHH^\dagger X^\dagger H^\dagger = HXX^\dagger H^\dagger = HH^\dagger = I$$

With similar computations, $(HXH)^\dagger HXH = I$. Thus HXH is unitary.

(b) We obtain successively for $|\varphi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$:

$$\begin{aligned}H|\varphi\rangle &= \frac{\alpha_0+\alpha_1}{\sqrt{2}}|0\rangle + \frac{\alpha_0-\alpha_1}{\sqrt{2}}|1\rangle\\ XH|\varphi\rangle &= \frac{\alpha_0+\alpha_1}{\sqrt{2}}|1\rangle + \frac{\alpha_0-\alpha_1}{\sqrt{2}}|0\rangle\\ HXH|\varphi\rangle &= \frac{\alpha_0+\alpha_1}{\sqrt{2}}\frac{|0\rangle-|1\rangle}{\sqrt{2}} + \frac{\alpha_0-\alpha_1}{\sqrt{2}}\frac{|0\rangle+|1\rangle}{\sqrt{2}} = \alpha_0|0\rangle - \alpha_1|1\rangle\end{aligned}$$

(c) The above gives

$$HXH|0\rangle = |0\rangle \qquad HXH|1\rangle = -|1\rangle \qquad HXH|+\rangle = |-\rangle \qquad HXH|-\rangle = |+\rangle$$

Exercise 4 *Production of Bell states*

(a) State (i) is a Bell entangled state (see below).
State (ii) is a product state $= |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.
State (iii) is an entangled state (cannot be written as $(\alpha_0|0\rangle + \alpha_1|1\rangle)\otimes(\beta_0|0\rangle + \beta_1|1\rangle)$).
State (iv) is a product state $= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\otimes\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.
State (v) is also a product state $= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\otimes\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

NB : An easy criterion for deciding when state

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

is a product state is $\det\begin{pmatrix}\alpha_{00} & \alpha_{01}\\ \alpha_{10} & \alpha_{11}\end{pmatrix} = 0$.

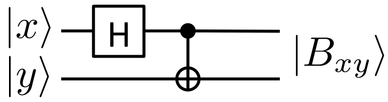
(b) A direct computation gives

$$\begin{aligned}(CNOT)(H \otimes I)\,|x\rangle \otimes |y\rangle &= (CNOT)H\,|x\rangle \otimes |y\rangle \\ &= (CNOT)\frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^x\,|1\rangle\right) \otimes |y\rangle \\ &= \frac{1}{\sqrt{2}}CNOT\,|0, y\rangle + \frac{(-1)^x}{\sqrt{2}}CNOT\,|1, y\rangle \\ &= \frac{1}{\sqrt{2}}\,|0, y\rangle + \frac{(-1)^x}{\sqrt{2}}\,|1, \overline{y}\rangle\end{aligned}$$

More explicitly, we enumerate all the cases :

$$\begin{aligned}(CNOT)(H \otimes I)\,|00\rangle &= (CNOT)\frac{1}{\sqrt{2}}\big(\,|00\rangle + |10\rangle\,\big) = \frac{1}{\sqrt{2}}\big(\,|00\rangle + |11\rangle\,\big) = |B_{00}\rangle \\ (CNOT)(H \otimes I)\,|01\rangle &= (CNOT)\frac{1}{\sqrt{2}}\big(\,|01\rangle + |11\rangle\,\big) = \frac{1}{\sqrt{2}}\big(\,|01\rangle + |10\rangle\,\big) = |B_{01}\rangle \\ (CNOT)(H \otimes I)\,|10\rangle &= (CNOT)\frac{1}{\sqrt{2}}\big(\,|00\rangle - |10\rangle\,\big) = \frac{1}{\sqrt{2}}\big(\,|00\rangle - |11\rangle\,\big) = |B_{10}\rangle \\ (CNOT)(H \otimes I)\,|11\rangle &= (CNOT)\frac{1}{\sqrt{2}}\big(\,|00\rangle - |11\rangle\,\big) = \frac{1}{\sqrt{2}}\big(\,|01\rangle - |10\rangle\,\big) = |B_{11}\rangle\end{aligned}$$

(c) The circuit corresponding to $|B_{xy}\rangle = (CNOT)(H \otimes I)|x\rangle \otimes |y\rangle$:



(d) The circuit corresponding to $|x\rangle \otimes |y\rangle = (H \otimes I)(CNOT)|B_{xy}\rangle$: