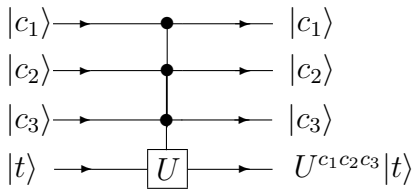
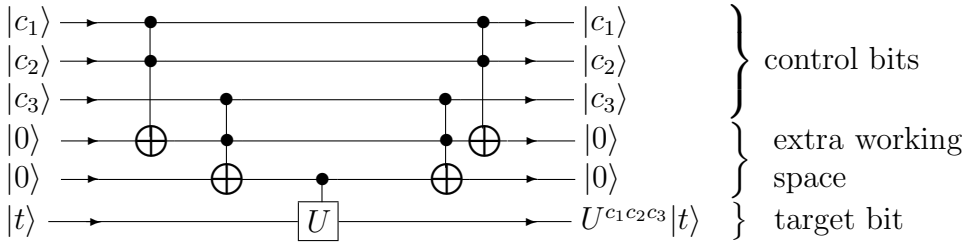
Exercise Set 3
Quantum Computation

Exercise 1 *Construction of a multi-control-U*

Verify that the multi-control-U :



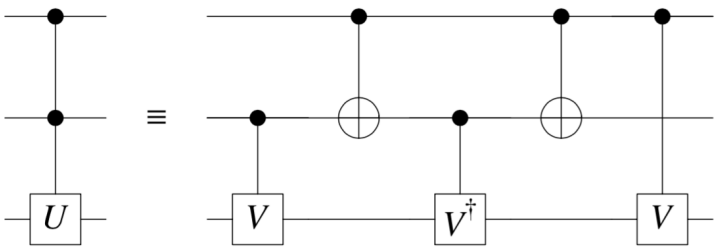
can be realized with the Toffoli gate (control-control-NOT) and a simple control-U.



Exercise 2 *Controlled-controlled-U*

In the last exercise, we have seen the construction of multi-controlled-U gate with the Toffoli gates and a simple controlled-U gate. This time, we are going to see a different construction for controlled-controlled-U gate.

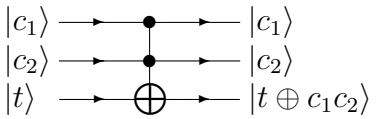
Let V be any quantum gate such that $V^2 = U$. Prove the following circuit identity.



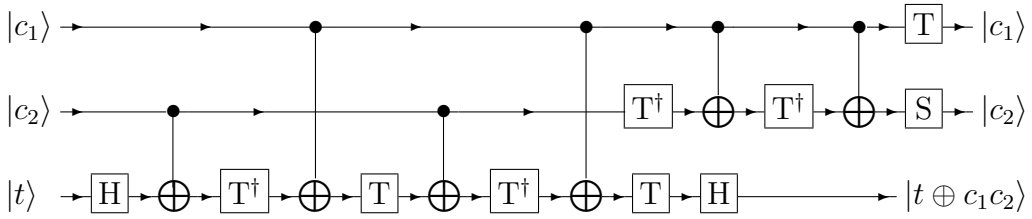
Explain in what sense this second construction is "not universal", whereas the one of the previous exercise is "universal".

Exercise 3 *Construction of the Toffoli gate from a control-NOT (! long computation !)*

Verify that the control-control-NOT also called Toffoli gate :



is equivalent to the following circuit made of $CNOT$, H, T and S



Hints : - Observe first that $(CNOT)\,|x\rangle \otimes |y\rangle = |x\rangle \otimes |x \oplus y\rangle$ can also be represented as $|x\rangle \otimes X^x\,|y\rangle$, where $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the NOT gate.

- Then it is also useful to compute first independently $XT^\dagger X$.