Exercise Set 3 : Solution
Quantum Computation

Exercise 1 *Construction of a multi-control-U*

We show the quantum state at each stage of the circuit.

$$\begin{aligned}
\text{Input}: &\ |c_1\rangle \otimes |c_2\rangle \otimes |c_3\rangle \otimes |0\rangle \otimes |0\rangle \otimes |t\rangle \\
\text{After the 1st Toffoli gate}: &\ |c_1\rangle \otimes |c_2\rangle \otimes |c_3\rangle \otimes |c_1 \cdot c_2\rangle \otimes |0\rangle \otimes |t\rangle \\
\text{After the 2nd Toffoli gate}: &\ |c_1\rangle \otimes |c_2\rangle \otimes |c_3\rangle \otimes |c_1 \cdot c_2\rangle \otimes |c_1 \cdot c_2 \cdot c_3\rangle \otimes |t\rangle \\
\text{After the contolled-}U\text{ gate}: &\ |c_1\rangle \otimes |c_2\rangle \otimes |c_3\rangle \otimes |c_1 \cdot c_2\rangle \otimes |c_1 \cdot c_2 \cdot c_3\rangle \otimes U^{c_1c_2c_3} |t\rangle \\
\text{After the 3rd Toffoli gate}: &\ |c_1\rangle \otimes |c_2\rangle \otimes |c_3\rangle \otimes |c_1 \cdot c_2\rangle \otimes |0\rangle \otimes U^{c_1c_2c_3} |t\rangle \\
\text{After the 4th Toffoli gate}: &\ |c_1\rangle \otimes |c_2\rangle \otimes |c_3\rangle \otimes |0\rangle \otimes |0\rangle \otimes U^{c_1c_2c_3} |t\rangle
\end{aligned}$$

Exercise 2 *Controlled-controlled-U*

Performing an analysis similar to the previous exercise, we observe that for an input state $|c_1\rangle \otimes |c_2\rangle \otimes |t\rangle$, the output of the circuit is

$$|c_1\rangle \otimes |c_2\rangle \otimes V^{c_1} (V^\dagger)^{c_1 \oplus c_2} V^{c_2} |t\rangle$$

which is equal to $|c_1\rangle \otimes |c_2\rangle \otimes |t\rangle$, unless $c_1 = c_2 = 1$, in which case the output is given by

$$|c_1\rangle \otimes |c_2\rangle \otimes V^2 |t\rangle = |c_1\rangle \otimes |c_2\rangle \otimes U |t\rangle$$

This second construction is not universal, as it requires to compute, for each gate U, the gate V such that $V^2 = U$.

Exercise 3 *Construction of the Toffoli gate from a control-NOT*

Using the first hint, we see that the circuit outputs the tensor product state $|\psi\rangle$ given by

$$|\psi\rangle = T |c_1\rangle \otimes SX^{c_1}T^\dagger X^{c_1} T^\dagger |c_2\rangle \otimes HTX^{c_1}T^\dagger X^{c_2} T X^{c_1} T^\dagger X^{c_2} H |t\rangle .$$

We then verify explicitly all the cases of c_1 and c_2. The calculation largely uses the fact that all the quantum gates here are unitary (*e.g.*, $TT^\dagger = T^\dagger T = I$); in particular, the gates X and H are involutory, *i.e.*, $X^2 = H^2 = I$.

For $c_1 = 0$, we have

$$\begin{aligned}
|\psi\rangle &= T |0\rangle \otimes ST^\dagger T^\dagger |c_2\rangle \otimes HTT^\dagger X^{c_2} TT^\dagger X^{c_2} H |t\rangle \\
&= |0\rangle \otimes |c_2\rangle \otimes H \left(TT^\dagger\right) \left(X^{c_2} \left(TT^\dagger\right) X^{c_2}\right) H |t\rangle = |0\rangle \otimes |c_2\rangle \otimes |t\rangle
\end{aligned}$$

For $c_1 = 1$ and $c_2 = 0$, let us follow the second hint :

$$XT^\dagger X = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & 1 \end{pmatrix} = e^{-i\pi/4} T \tag{1}$$

and use this to compute

$$\begin{aligned}
|\psi\rangle &= T|1\rangle \otimes SXT^\dagger XT^\dagger |0\rangle \otimes HTXT^\dagger TXT^\dagger H |t\rangle \\
&= e^{i\pi/4}|1\rangle \otimes S\left(XT^\dagger X\right)T^\dagger |0\rangle \otimes H\left(T\left(X\left(T^\dagger T\right)X\right)T^\dagger\right)H|t\rangle \\
&= e^{i\pi/4}|1\rangle \otimes e^{-i\pi/4} STT^\dagger |0\rangle \otimes |t\rangle \\
&= e^{i\pi/4}|1\rangle \otimes e^{-i\pi/4}|0\rangle \otimes |t\rangle = |1\rangle \otimes |0\rangle \otimes |t\rangle
\end{aligned}$$

Finally, for $c_1 = c_2 = 1$, we compute, using repeatedly (1) :

$$\begin{aligned}
|\psi\rangle &= T|1\rangle \otimes SXT^\dagger XT^\dagger |1\rangle \otimes HTXT^\dagger XTXT^\dagger XH|t\rangle \\
&= e^{i\pi/4}|1\rangle \otimes e^{-i\pi/4} STT^\dagger |1\rangle \otimes e^{-i\pi/2} HT^4 H|t\rangle \\
&= e^{i\pi/4}|1\rangle \otimes e^{i\pi/4}|1\rangle \otimes e^{-i\pi/2} X|t\rangle
\end{aligned}$$

as

$$T^4 = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and therefore

$$HT^4H = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X$$

Finally, this gives

$$|\psi\rangle = |1\rangle \otimes |1\rangle \otimes |\bar{t}\rangle$$

as expected.