Homework 6

Exercise 1. Let $(X_n,\, n \geq 1)$ be independent random variables such that $X_n \sim \text{Bern}(1 - \frac{1}{(n+1)^\alpha})$, where $\alpha > 0$.

Let us also define $Y_n = \prod_{j=1}^n X_j$ for $n \geq 1$.

a) What minimal condition on the parameter $\alpha > 0$ ensures that $Y_n \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} 0$?

Hint: Use the approximation $1 - x \simeq \exp(-x)$ for x small.

b) Under the same condition as that found in a), does it also hold that $Y_n \underset{n\to\infty}{\overset{L^2}{\to}} 0$?

c) Under the same condition as that found in a), does it also hold that $Y_n \underset{n\to\infty}{\to} 0$ almost surely ?

Hint: If $Y_n = 0$, what can you deduce on Y_m for $m \geq n$?

Exercise 2. a) Show that if $(A_n,\, n \geq 1)$ are *independent* events in $\mathcal{F}$ and $\sum_{n\geq 1} \mathbb{P}(A_n) = \infty$, then

$$\mathbb{P}\Big(\bigcup_{n\geq 1} A_n\Big) = 1$$

Hints: - Start by observing that the statement is equivalent to $\mathbb{P}\left(\bigcap_{n\geq 1} A_n^c\right) = 0$.
- Use the inequality $1 - x \leq e^{-x}$, valid for all $x \in \mathbb{R}$.

b) From the same set of assumptions, reach the following stronger conclusion with a little extra effort:

$$\mathbb{P}(\{\omega \in \Omega : \omega \in A_n \text{ infinitely often}\}) = \mathbb{P}\Big(\bigcap_{N\geq 1} \bigcup_{n\geq N} A_n\Big) = 1$$

which is actually the statement of the *second Borel-Cantelli lemma.*

c) Let $(X_n,\, n \geq 1)$ be a sequence of *independent* random variables such that for some $\varepsilon > 0$, $\sum_{n\geq 1} \mathbb{P}(\{|X_n| \geq \varepsilon\}) = +\infty$. What can you conclude on the almost sure convergence of the sequence X_n towards the limiting value 0?

d) Let $(X_n,\, n \geq 1)$ be a sequence of independent random variables such that $\mathbb{P}(\{X_n = n\}) = p_n = 1 - \mathbb{P}(\{X_n = 0\})$ for $n \geq 1$. What minimal condition on the sequence $(p_n, n \geq 1)$ ensures that

d1) $X_n \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} 0$? d2) $X_n \underset{n\to\infty}{\overset{L^2}{\to}} 0$? d3) $X_n \underset{n\to\infty}{\to} 0$ almost surely?

e) Let $(Y_n,\, n \geq 1)$ be a sequence of independent random variables such that $Y_n \sim \text{Cauchy}(\lambda_n)$ for $n \geq 1$. What minimal condition on the sequence $(\lambda_n,\, n \geq 1)$ ensures that

e1) $Y_n \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} 0$? e2) $Y_n \underset{n\to\infty}{\overset{L^2}{\to}} 0$? e3) $Y_n \underset{n\to\infty}{\to} 0$ almost surely?

Exercise 3*. (extended law of large numbers)
Let $(\mu_n,\, n \geq 1)$ be a sequence of real numbers such that

$$\lim_{n\to\infty} \frac{\mu_1 + \ldots + \mu_n}{n} = \mu \in \mathbb{R}$$

Let $(X_n,\, n \geq 1)$ be a sequence of square-integrable random variables such that

$$\mathbb{E}(X_n) = \mu_n, \quad \forall n \geq 1 \quad \text{and} \quad \mathrm{Cov}(X_n, X_m) \leq C_1 \exp(-C_2\,|m-n|), \quad \forall m, n \geq 1.$$

for some constants $C_1, C_2 > 0$ (the random variables X_n are said to be *weakly* correlated). Let finally $S_n = X_1 + \ldots + X_n$.

a) Show that

$$\frac{S_n}{n} \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} \mu$$

b) Is it also true that

$$\frac{S_n}{n} \underset{n\to\infty}{\to} \mu \quad \text{almost surely?}$$

In order to check this, you need to go through the proof of the strong law of large numbers made in class. Does that proof need the fact that the random variables X_n are independent?

c) Application to auto-regressive processes: Let $(Z_n,\, n \geq 1)$ be a sequence of i.i.d. $\sim \mathcal{N}(0,1)$ random variables, $x, a \in \mathbb{R}$ and $(X_n,\, n \geq 1)$ be the sequence of random variables defined recursively as

$$X_1 = x, \quad X_{n+1} = aX_n + Z_{n+1}, \quad n \geq 1$$

For what values of $x, a \in \mathbb{R}$ does the sequence $(X_n,\, n \geq 1)$ satisfy the assumptions made in a)? Compute μ in this case.

Exercise 4. (another extension of the weak law of large numbers)
Let $(X_n,\, n \geq 1)$ be a sequence of i.i.d. square-integrable random variables such that $\mathbb{E}(X_1) = \mu \in \mathbb{R}$ and $\mathrm{Var}(X_1) = \sigma^2 > 0$.

Let $(T_n,\, n \geq 1)$ be another sequence of random variables, independent of the sequence $(X_n,\, n \geq 1)$, with all T_n taking values in the set of natural numbers $\mathbb{N}^* = \{1, 2, 3, \ldots\}$. Define

$$p_k^{(n)} = \mathbb{P}(\{T_n = k\}) \quad \text{for } n, k \geq 1 \qquad \left(\text{so } \sum_{k\geq 1} p_k^{(n)} = 1 \quad \forall n \geq 1\right)$$

a) Find a sufficient condition on the numbers $p_k^{(n)}$ guaranteeing that

$$\frac{X_1 + \ldots + X_{T_n}}{T_n} \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} \mu \tag{1}$$

Hint: You should use the law of total probability here: if A is an event and the events $(B_k,\, k \geq 1)$ form a partition of Ω, then:

$$\mathbb{P}(A) = \sum_{k\geq 1} \mathbb{P}(A \,|\, B_k)\, \mathbb{P}(B_k)$$

b) Apply the above criterion to the following case: each T_n is the sum of two independent geometric random variables $G_{n1} + G_{n2}$, where both G_n are distributed as

$$\mathbb{P}(\{G_n = k\}) = q_n^{k-1}\,(1 - q_n) \quad k \geq 1$$

where $0 < q_n < 1$.

b1) Compute first the distribution of T_n, as well as $\mathbb{E}(T_n)$, for each $n \geq 1$.

b2) What condition on the sequence $(q_n,\, n \geq 1)$ ensures that conclusion (1) holds?

Hint: Solving question b1) above may help you guessing what the answer to b2) should be.