Exercise Set 5
Quantum Computation

Exercise 1 *Square-root of the NOT gate*

The aim of the present exercise is to compute, for a given one-qubit gate U, a corresponding gate V such that $V^2 = U$ (cf. Ex 2, Hw 3) in the particular case where $U = X$ (the NOT gate). Here is first a description of the generic procedure.

First observe that since a one-qubit gate U is a 2×2 unitary matrix ($UU^\dagger = U^\dagger U = I$), it is in particular a *normal* matrix satisfying $UU^\dagger = U^\dagger U$. The *spectral theorem* then asserts that such a U is *unitarily diagonalizable*, i.e., there exists Λ a 2×2 diagonal matrix (with possibly complex entries) and W another 2×2 unitary matrix, such that $U = W \, \Lambda \, W^\dagger$.

In order to compute Λ and W, it suffices to compute the two solutions to the eigenvalue-eigenvector equation:

$$U w^{(i)} = \lambda_i \, w^{(i)}, \quad i = 0, 1$$

with the added constraint that $(w^{(i)})^\dagger w^{(j)} = \delta_{i,j}$. Then $\Lambda = \mathrm{diag}(\lambda_0, \lambda_1)$ and $W = (w^{(0)}, w^{(1)})$, i.e., $w^{(i)}$ is the i-th column of W.

Finally, consider $V = W \sqrt{\Lambda} \, W^\dagger$, where $\sqrt{\Lambda} = \mathrm{diag}(\sqrt{\lambda_0}, \sqrt{\lambda_1})$, with square roots being taken in the complex plane $\mathbb{C}$ (! two options for each of them !). You can check that $V^2 = U$.

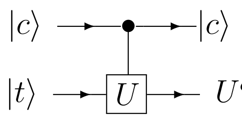
(a) Compute the 2×2 matrices Λ and W corresponding to $U = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(b) Deduce an explicit expression for a matrix V such that $V^2 = X$. Check now directly that $V^2 = X$.

(c) Is V also unitary? Justify your answer.

Exercise 2 *SWAP · Controlled-U · SWAP*

Let $U = \begin{pmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{pmatrix}$ be a unitary matrix. The Controlled-U gate is given by the following:



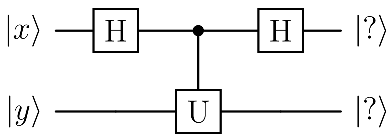
$$U^c|t\rangle = \begin{cases} |t\rangle \text{ if } c = 0 \\ U|t\rangle \text{ if } c = 1 \end{cases}$$

(a) Compute the matrix representation of the Controlled-U gate.

(b) Draw the circuit for SWAP· Controlled-U ·SWAP and compute its matrix representation.

Exercise 3 *Another (small) quantum algorithm*

Let U be a unitary matrix and $|u\rangle$ be an eigenvector of U, that is, $U|u\rangle = \exp(2\pi i\varphi)|u\rangle$. Consider the following circuit:



(a) Compute the output corresponding to the input $|0\rangle \otimes |u\rangle$.

(b) Compute the probability of observing the first qubit in state $|0\rangle$, respectively $|1\rangle$, at the output of this circuit.

(c) Assume replacing U by U^k with k integer in the above circuit. Let $\varphi = 0, \varphi_1\varphi_2...\varphi_t$ be the binary expansion of $0 < \varphi < 1$. How then to choose k to determine the least significant bit φ_t with a single measurement?