Exercise Set 5: Solution
Quantum Computation

Exercise 1 *Square-root of the NOT gate*

(a) The solutions to the eigenvalue-eigenvector equation are:

$$\lambda_0 = 1, \quad w^{(0)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \lambda_1 = -1, \quad w^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

so $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $W = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

(b) We deduce from the above that a possible V is

$$V = W\sqrt{\Lambda}\,W^\dagger = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$$

and indeed, one can check directly that $V^2 = U$.

(c) Yes, V is also unitary, as $V^\dagger = \frac{1}{2} \begin{pmatrix} 1-i & 1+i \\ 1+i & 1-i \end{pmatrix}$ and $VV^\dagger = I$.

Remark: Please note that we have already encountered $W = H$, $\Lambda = Z$ and $\sqrt{\Lambda} = S$!

Exercise 2 *SWAP · Controlled-U · SWAP*

(a) The action of the Controlled-U gate is

$$\text{Controlled-U}\,(|c\rangle \otimes |t\rangle) = |c\rangle \otimes U^c\,|t\rangle$$

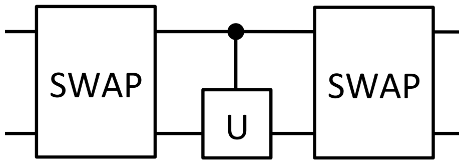
so

$$\text{Controlled-U}\,(|0\rangle \otimes |t\rangle) = |0\rangle \otimes |t\rangle \quad \text{and} \quad \text{Controlled-U}\,(|1\rangle \otimes |t\rangle) = |1\rangle \otimes U\,|t\rangle$$

from which we deduce the matrix representation of the Controlled-U gate in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$:

$$\text{Controlled-U} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & U_{00} & U_{01} \\ 0 & 0 & U_{10} & U_{11} \end{pmatrix}$$

(b) The circuit for SWAP $\cdot$ Controlled-U $\cdot$ SWAP is:



Now, SWAP has the following matrix representation in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$:

$$\text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We therefore obtain

$$\begin{aligned} &\text{SWAP} \cdot \text{Controlled-}U \cdot \text{SWAP} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & U_{00} & U_{01} \\ 0 & 0 & U_{10} & U_{11} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & U_{00} & 0 & U_{01} \\ 0 & 0 & 1 & 0 \\ 0 & U_{10} & 0 & U_{11} \end{pmatrix} \end{aligned}$$

Exercise 3 *Another (small) quantum algorithm*

(a)

$$H|0\rangle \otimes |u\rangle = \frac{1}{\sqrt{2}}|0\rangle \otimes |u\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes |u\rangle$$

$$CU\,H|0\rangle \otimes |u\rangle = \frac{1}{\sqrt{2}}|0\rangle \otimes |u\rangle + \frac{1}{\sqrt{2}}e^{2\pi i\varphi}|1\rangle \otimes |u\rangle$$

$$\begin{aligned} H\,CU\,H|0\rangle \otimes |u\rangle &= \frac{1}{2}(|0\rangle + |1\rangle) \otimes |u\rangle + \frac{e^{2\pi i\varphi}}{2}(|0\rangle - |1\rangle) \otimes |u\rangle \\ &= \frac{1 + e^{2\pi i\varphi}}{2}|0\rangle \otimes |u\rangle + \frac{1 - e^{2\pi i\varphi}}{2}|1\rangle \otimes |u\rangle \\ &= e^{\pi i\varphi}(\cos \pi\varphi\,|0\rangle \otimes |u\rangle - i\sin \pi\varphi\,|1\rangle \otimes |u\rangle) \end{aligned}$$

(b)

$$\text{Prob}(0) = \cos^2 \pi\varphi \quad \text{and} \quad \text{Prob}(1) = \sin^2 \pi\varphi$$

(c) If we apply U^k instead of U, we find the output:

$$e^{i\pi k\varphi}(\cos(\pi k\varphi)|0\rangle \otimes |u\rangle - i\sin(\pi k\varphi)|1\rangle \otimes |u\rangle)$$

If $\varphi = \frac{\varphi_1}{2} + \frac{\varphi_2}{2^2} + \cdots + \frac{\varphi_{t-1}}{2^{t-1}} + \frac{\varphi_t}{2^t}$, then taking $k = 2^{t-1}$, we observe state 0 with probability

$$\text{Prob}(0) = \cos^2\left(\pi\varphi_{t-1} + \frac{\pi\varphi_t}{2}\right) = \cos^2\left(\frac{\pi\varphi_t}{2}\right) = \begin{cases} 1 & \text{if } \varphi_t = 0 \\ 0 & \text{if } \varphi_t = 1 \end{cases}$$