Final exam: Solutions

Exercise 1. Quiz. (20 points)
Answer each yes/no question below (2 pts) and provide a short justification for your answer (2 pts).

a) Does there exist a sequence of independent random variables $(X_n,\, n \geq 1)$ such that the sequence of σ-fields $(\mathcal{F}_n,\, n \geq 1)$ defined as

$$\mathcal{F}_n = \sigma(X_1 + \ldots + X_n), \quad n \geq 1$$

is a filtration, and moreover, $\mathcal{F}_n \neq \mathcal{F}_{n+1}$ for every $n \geq 1$? If yes, exhibit such a sequence of random variables; if no, explain why such a sequence cannot exist.

Answer: Yes. Consider for example X_n independent with $\mathbb{P}(\{X_n = 2^{-n}\}) = \mathbb{P}(\{X_n = 0\}) = \frac{1}{2}$.

b) Let X, Z be two independent random variables such that $\mathbb{P}(\{X = +1\}) = \mathbb{P}(\{X = +2\}) = \frac{1}{2}$ and $\mathbb{P}(\{Z = -1\}) = \mathbb{P}(\{Z = 0\}) = \frac{1}{2}$. Let also $Y = X \cdot Z$. Does it hold that $t \mapsto F_Y(t)$ has four jumps (i.e., four values of t where it is not continuous) ? Justify your answer.

Answer: No. F_Y has actually three jumps in $t = -2, -1, 0$ (of sizes $\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{2}$, respectively).

c) Let X_1, X_2 be two random variables such that $X_1 \sim \mathcal{N}(0, \sigma_1^2)$, $X_2 \sim \mathcal{N}(0, \sigma_2^2)$ and

$$\mathbb{E}(\exp(it_1 X_1 + it_2 X_2)) = \exp\left(-\frac{(\sigma_1\, t_1 + \sigma_2\, t_2)^2}{2}\right), \quad \forall t_1, t_2 \in \mathbb{R}$$

Does it hold that X_1 and X_2 are independent ? Justify your answer.

Answer: No. In this case, it holds that $X_2 = \frac{\sigma_2}{\sigma_1} X_1$. Independence would hold if the following inequality were true:

$$\mathbb{E}(\exp(it_1 X_1 + it_2 X_2)) = \exp\left(-\frac{(\sigma_1\, t_1)^2 + (\sigma_2\, t_2)^2}{2}\right) = \phi_{X_1}(t_1) \cdot \phi_{X_2}(t_2)$$

d) Let X_1, X_2 be two independent random variables taking values in $\{-1, +1\}$ and such that $\mathbb{P}(\{X_1 = +1\}) = \mathbb{P}(\{X_1 = -1\}) = \frac{1}{2}$. Does it hold that

$$\mathbb{E}(\,|X_1 + X_2|\mid X_2) = |X_2| \quad ?$$

Justify your answer.

Answer: Yes: $\mathbb{E}(\,|X_1 + X_2| \mid X_2) = \varphi(X_2)$, where $\varphi(y) = \mathbb{E}(|X_1 + y|) = 1 = |y|$ for both $y = +1$ and $y = -1$.

e) Do there exist real numbers $p, q \in [0, 1]$ and random variables X_1, X_2, X_3 taking values in $\{-1, +1\}$ satisfying the following properties ?

- $\mathbb{P}(\{X_1 = x_1, X_2 = x_2, X_3 = x_3\}) = \begin{cases} p & \text{if } x_1 = x_2 = x_3 \\ q & \text{otherwise} \end{cases}$

- $\mathbb{E}(X_j\,X_k) = -\frac{1}{2}$ for every $j,k \in \{1,2,3\}$ with $j < k$.

If yes, exhibit such random variables X_1, X_2, X_3, along with the corresponding values of p and q; if no, explain why they cannot exist.

Answer: No. Here is why: on the one hand, we must have $2p + 6q = 1$, so $p + 3q = \frac{1}{2}$; on the other hand, we must also have

$$-\frac{1}{2} = \mathbb{E}(X_1 X_2) = \mathbb{P}(\{X_1 = X_2\}) - \mathbb{P}(\{X_1 \neq X_2\}) = 2p + 2q - 4q = 2(p-q) \quad \text{so} \quad p - q = -\frac{1}{4}$$

Substracting the two equations, we obtain $4q = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, so $q = \frac{3}{16}$, but then $p = q - \frac{1}{4} = -\frac{1}{16}$, which is impossible.

Note: There do exist identically distributed random variables X_1, X_2, X_3 such that $\mathbb{E}(X_j\,X_k) = -\frac{1}{2}$ for every $j,k \in \{1,2,3\}$ with $j < k$, but these must take values in a larger set than $\{-1,+1\}$.

Exercise 2. (15 points)

Note: For this exercise, the following facts might help:

- For $s,t \in \mathbb{R}$, $\cosh(s) = \frac{\exp(s)+\exp(-s)}{2}$ and $\cos(t) = \frac{\exp(it)+\exp(-it)}{2}$
- For $x \in \mathbb{R}$ and $n \geq 1$, $(1 + \frac{x}{n})^n \leq \exp(x)$; also, $\lim_{n\to\infty}(1 + \frac{x}{n})^n = \exp(x)$

For a given value of $n \geq 1$, let $(X_1^{(n)}, \ldots, X_n^{(n)})$ be i.i.d. random variables such that

$$\mathbb{P}(\{X_1^{(n)} = +1\}) = \mathbb{P}(\{X_1^{(n)} = -1\}) = \frac{1}{2n} \quad \text{and} \quad \mathbb{P}(\{X_1^{(n)} = 0\}) = 1 - \frac{1}{n}$$

Let also $S_n = X_1^{(n)} + \ldots + X_n^{(n)}$.

a) Compute $\mathbb{E}(S_n)$ and $\mathrm{Var}(S_n)$ for $n \geq 1$.

Answer: $\mathbb{E}(S_n) = 0$ and $\mathrm{Var}(S_n) = n \cdot \frac{1}{n} = 1$.

b) Using Chebyshev's inequality with $\varphi(x) = \exp(sx)$, $s > 0$, show that for every $c > 0$

$$\mathbb{P}(\{|S_n| \geq c\,\log(n) \quad \text{infinitely often}\}) = 0$$

Hint: There is no need here to optimize over s your upper bound on the probability: for each value of $c > 0$, you just need to find an appropriate value of $s > 0$ that allows to conclude.

Answer: Since S_n is symmetrically distributed, it is enough to show $\mathbb{P}(\{S_n \geq c\,\log(n) \text{ i.o.}\}) = 0$. Chebyshev's inequality gives

$$\mathbb{P}(\{S_n \geq c\,\log(n)\}) \leq \frac{\mathbb{E}(\exp(sS_n))}{n^{cs}}$$

and by the hint at the beginning of the exercise

$$\mathbb{E}(\exp(sS_n)) = \left(1 + \frac{\cosh(s) - 1}{n}\right)^n \leq \exp(\cosh(s) - 1)$$

so

$$\mathbb{P}(\{S_n \geq c\,\log(n)\}) \leq \frac{\exp(\cosh(s)-1)}{n^{cs}}$$

Choosing then $s = 2/c$ ensures that $\mathbb{P}(\{S_n \geq c\,\log(n)\}) = O(1/n^2)$. Since this is summable over n, the Borel-Cantelli lemma allows to conclude.

c) Show that the sequence of random variables $(S_n,\, n \geq 1)$ converges in distribution.

Hint: You may use here the fact (not seen in class) that if a sequence of characteristic functions converges to a limit which is a continuous function, then the corresponding sequence of random variables converges in distribution (no need to compute the limiting distribution).

Answer: By the hint, we have

$$\mathbb{E}(\exp(itS_n)) = \left(1 + \frac{\cos(t)-1}{n}\right)^n \underset{n\to\infty}{\to} \exp(\cos(t)-1).$$

which is a continuous function. This allows us to conclude that the sequence $(S_n,\, n \geq 1)$ converges in distribution.

d) Can you tell whether the limiting distribution in part c) is discrete or continuous ? Justify your answer.

Hint: Again, there is no need to compute explicitly the limiting distribution here.

Answer: S_n can only take integer values for all values of n, so the limiting distribution must be discrete. Another possible argument is that the limiting characteristic function is periodic (and therefore does not go to zero as t goes to $\pm\infty$).

Exercise 3. (17 points) Let $N \geq 1$ be a fixed integer and $Z = (Z_1, \ldots, Z_N)$ be a random vector composed of i.i.d. $\mathcal{N}(0,1)$ random variables. Let also

$$\|Z\| = \sqrt{\sum_{n=1}^N Z_n^2}, \quad Y_n = \frac{Z_n}{\|Z\|}, \quad \text{for } 1 \leq n \leq N$$

Hint: For this exercise, you may use the rather remarkable fact that *conditioned on* $\|Z\|$, the vector $(Z_1, \ldots, Z_N)$ is uniformly distributed on the N-dimensional sphere of radius $\|Z\|$. This implies in particular that for every n, the vector $(Z_1, \ldots, -Z_n, \ldots, Z_N)$ is also uniformly distributed on the N-dimensional sphere of radius $\|Z\|$.

a) Compute $\mathbb{E}(Y_n)$, $\mathrm{Var}(Y_n)$ and $\mathbb{E}(Y_n\, Y_m)$ for $1 \leq n, m \leq N$ with $n \neq m$.

Answer: By the hint, $\mathbb{E}(Z_n \,|\, \|Z\|) = \mathbb{E}(-Z_n \,|\, \|Z\|) = 0$. Therefore,

$$\mathbb{E}(Y_n) = \mathbb{E}\left(\frac{Z_n}{\|Z\|}\right) = \mathbb{E}\left(\mathbb{E}\left(\frac{Z_n}{\|Z\|} \,\middle|\, \|Z\|\right)\right) = \mathbb{E}\left(\frac{\mathbb{E}(Z_n \,|\, \|Z\|)}{\|Z\|}\right) = 0$$

and similarly, $\mathbb{E}(Z_n\, Z_m \,|\, \|Z\|) = \mathbb{E}(-Z_n Z_m \,|\, \|Z\|) = 0$, so $\mathbb{E}(Y_n\, Y_m) = 0$.

Consequently, $\mathrm{Var}(Y_n) = \mathbb{E}(Y_n^2)$. Note that $(Y_1, \dots, Y_n)$ are identically distributed and

$$\sum_{n=1}^{N} \mathbb{E}(Y_n^2) = \mathbb{E}\left(\frac{\sum_{n=1}^N Z_n^2}{\|Z\|^2}\right) = \mathbb{E}\left(\frac{\sum_{n=1}^N Z_n^2}{\sum_{n=1}^N Z_n^2}\right) = 1$$

so $\mathbb{E}(Y_n^2) = \frac{1}{N}$ for all n.

Let now $M_0 = 0$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $M_n = \sum_{j=1}^n Y_j$, $\mathcal{F}_n = \sigma(\|Z\|, Y_1, \dots, Y_n)$ for $1 \leq n \leq N$.

b) Compute $\mathbb{E}(M_n)$ and $\mathrm{Var}(M_n)$ for $1 \leq n \leq N$.

Answer: $\mathbb{E}(M_n) = \sum_{m=1}^n \mathbb{E}(Y_m) = 0$ and

$$\mathrm{Var}(M_n) = \mathbb{E}(M_n^2) = \sum_{m=1}^{n} \mathbb{E}(Y_m^2) + \sum_{m \neq \ell} \mathbb{E}(Y_m\, Y_\ell) = \frac{n}{N}$$

as $\mathbb{E}(Y_n^2) = \frac{1}{N}$ and $\mathbb{E}(Y_m\, Y_\ell) = 0$.

c) Show that $\mathcal{F}_n = \sigma(\|Z\|, Z_1, \dots, Z_n)$ for every $1 \leq n \leq N$.

Answer: $(\|Z\|, Z_1, \dots, Z_N) \mapsto (\|Z\|, \frac{Z_1}{\|Z\|}, \dots, \frac{Z_N}{\|Z\|}) = (\|Z\|, Y_1, \dots, Y_N)$ is a 1-to-1 transformation. Thus, the statement holds.

d) Show that $(M_n,\, 0 \leq n \leq N)$ is martingale with respect to $(\mathcal{F}_n,\, 0 \leq n \leq N)$.

Answer: It is easy to see that M_n is integrable, as it is the sum of bounded random variables, and $\mathcal{F}_n$-measurable. Moreover,

$$\mathbb{E}(M_n \,|\, \mathcal{F}_{n-1}) = M_{n-1} + \mathbb{E}\left(\frac{Z_n}{\|Z\|} \,\middle|\, \mathcal{F}_{n-1}\right) = M_{n-1} + \frac{\mathbb{E}(Z_n|\mathcal{F}_{n-1})}{\|Z\|}$$

Again, by the same reasoning as in part a), $\mathbb{E}(Z_n \,|\, \mathcal{F}_{n-1}) = \mathbb{E}(-Z_n \,|\, \mathcal{F}_{n-1}) = 0$, hence M is a martingale.

e) Using a theorem seen in the course (not forgetting to check its assumptions), show that

$$\mathbb{P}(\{|M_n - M_0| \geq nt\}) \leq 2\, \exp(-nt^2/2), \quad \text{for every } 1 \leq n \leq N \text{ and } t > 0$$

Answer: We use Azuma's inequality. Since $|M_n - M_{n-1}|^2 = \frac{Z_n^2}{\|Z\|^2} \leq 1$, $|M_n - M_{n-1}| \leq 1$ as well. Hence, we directly apply Azuma's inequality to obtain the result.

Exercise 4. (18 points + BONUS 5 points) Let us consider the process $(M_n,\, n \geq 0)$ defined recursively as:

$$M_0 = M_1 = 1, \quad M_{n+1} = \begin{cases} M_n + M_{n-1} & \text{with probability } \frac{1}{2} \\ \\ M_n - M_{n-1} & \text{with probability } \frac{1}{2} \end{cases} \quad \text{for } n \geq 1$$

a) Show that $(M_n,\, n \geq 0)$ is a martingale with respect to its natural filtration $(\mathcal{F}_n,\, n \geq 0)$.

Answer: First, observe that M_n takes a finite number of values for a given value of n, so it trivially holds that $\mathbb{E}(|M_n|) < +\infty$ for every $n \geq 0$. Besides, the martingale property holds trivially between $n = 0$ and $n = 1$, and for $n \geq 1$, we have

$$\mathbb{E}(M_{n+1} \,|\, \mathcal{F}_n) = \frac{1}{2}\,(M_n + M_{n-1}) + \frac{1}{2}\,(M_n - M_{n-1}) = M_n$$

b) Does it also hold that $\mathbb{E}(M_{n+1} \,|\, M_n) = M_n$ for every $n \geq 0$? Justify your answer.

Answer: Yes: By the towering property of conditional expectation and part a), we have

$$\mathbb{E}(M_{n+1} \,|\, M_n) = \mathbb{E}(\mathbb{E}(M_{n+1} \,|\, \mathcal{F}_n) \,|\, M_n) = \mathbb{E}(M_n \,|\, M_n) = M_n$$

c) Compute the distribution of M_4.

Hint: It is highly advised to draw carefully the binary tree leading to the possible values of M_4.

Answer: $\mathbb{P}(\{M_4 = +5\}) = \mathbb{P}(\{M_4 = +3\}) = \frac{1}{8}$ and $\mathbb{P}(\{M_4 = +1\}) = \mathbb{P}(\{M_4 = -1\}) = \frac{3}{8}$.

d) Compute recursively the sequence of numbers $f(n) = \mathbb{E}(M_n^2)$, for $n \geq 0$. What is this sequence ?

Answer: We find recursively that

$$\mathbb{E}(M_{n+1}^2 \,|\, \mathcal{F}_n) = \frac{1}{2}(M_n + M_{n-1})^2 + \frac{1}{2}\,(M_n - M_{n-1})^2 = M_n^2 + M_{n-1}^2$$

so taking expectations, we obtain $f(n+1) = f(n) + f(n-1)$ for $n \geq 1$; this is the Fibonacci sequence (so the value of $f(n)$ is growing exponentially with n).

e) Are the conditions of the first martingale convergence theorem satisfied ?

Answer: No: By part d), $\sup_{n \geq 0} \mathbb{E}(M_n^2) = +\infty$.

f) Does there exist a random variable M_∞ such that $M_n \underset{n\to\infty}{\to} M_\infty$ almost surely ? Justify your answer.

Hint: Consider the values that M_n can take when $n = 2\,(\text{mod } 3)$ and $n \neq 2\,(\text{mod } 3)$.

Answer: No: If the sequence $(M_n,\, n \geq 0)$ were to converge almost surely, then it would hold that for a given value of ω, $M_n(\omega)$ is equal to a fixed value K for large n, as $M_n(\omega)$ only takes integer values. But $M_n(\omega)$ oscillates between odd and even values (more precisely, $M_n(\omega)$ is even if and only if $n = 2\,(\text{mod } 3)$).

BONUS g) For a fixed value of $n \geq 3$, what is the value of $\max_{\omega\in\Omega} M_n(\omega)$? And what is the value of $\min_{\omega\in\Omega} M_n(\omega)$? Justify.

Answer: $\max_{\omega\in\Omega} M_n(\omega) = f(n)$, which is obtained by always following the path in the "+" direction and $\min_{\omega\in\Omega} M_n(\omega) = -f(n-3)$, which is obtained by following the path:

$$\ldots, \quad M_{n-3} = f(n-3), \quad M_{n-2} = f(n-2), \quad M_{n-1} = M_{n-2} - M_{n-3} = f(n-2) - f(n-3)$$

and finally: $M_n = M_{n-1} - M_{n-2} = -f(n-3)$