Homework 1

Traitement Quantique de l'Information

In this exercise we illustrate a toy model description of the "orbital" state of a photon and its manipulation by mirrors. We suppose that a photon can travel only along the "vertical" and "horizontal" directions and represent its general state by a quantum bit

$$\alpha |H\rangle + \beta |V\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

where $|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Recall that $\alpha, \beta \in \mathbb{C}$ and $\alpha^* \alpha + \beta^* \beta = 1$.

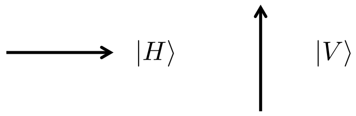


Figure 1: Possible preparation directions and states

1) Write down the "Bra" in Dirac and usual vector notations associated to the "Ket" $\alpha |H\rangle + \beta |V\rangle$.
2) Compute the scalar product (or Bracket) for the two kets $\alpha |H\rangle + \beta |V\rangle$ and $\gamma |H\rangle + \delta |V\rangle$ in Dirac and vector notations. In particular, check $\langle H|V\rangle = \langle V|H\rangle = 0$ and $\langle H|H\rangle = \langle V|V\rangle = 1$.
3) A mirror like Figure 2 makes the transitions $|H\rangle \to i |V\rangle$ and $|V\rangle \to i |H\rangle$. In quantum

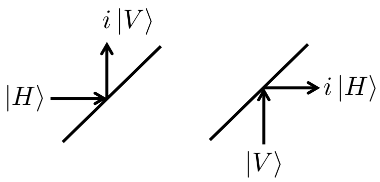
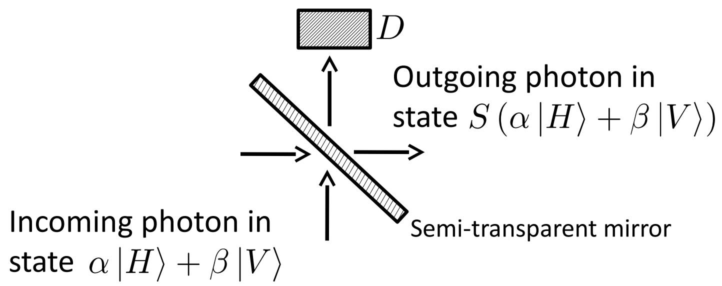


Figure 2: Mirror operations (perfect reflection)

physics the mirror operation is described by a matrix $R = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. Verify $R^\dagger R = R R^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ where $R^\dagger = R^{\top,*}$ (transpose and complex conjugate).
A photon in the state $\alpha |H\rangle + \beta |V\rangle$ is incident on the mirror and is reflected. Compute the output state $R(\alpha |H\rangle + \beta |V\rangle)$ in Dirac and vector notations. Make a picture of the photon and mirror.

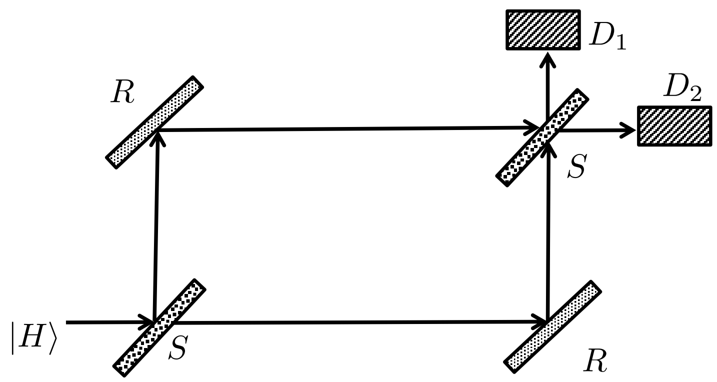
4) A semi-transparent mirror is described by a matrix $S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$. Verify $S^\dagger S = SS^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Compute $S\,|H\rangle$, $S\,|V\rangle$ and $S\,(\alpha\,|H\rangle + \beta\,|V\rangle)$. Make pictures analogous to Figure 2 to get an intuition of these operations.

5) Consider the following experiment where D is a photo-detector.



Here we will assume α and β are real. Compute the probability of detecting a photon in D.

6) Now we consider the following set-up which constitutes the Mach–Zehnder interferometer. It is constituted of two semi-transparent mirrors, two perfectly reflecting mirrors and two photodectectors. The "paths" of the photon are depicted on the figure.



The incoming photon has the state $|H\rangle$. Compute the final state just after the second semi-transparent mirror before the photodetectors. Then compute the probability of detecting the photon in the photodetectors D_1 and D_2. *What would you expect to find if the photons were "classical balls"?*