
Homework 1 Solution
Traitement Quantique de l'Information

1) The “Ket” and the associated Dirac or usual vector notations are:

- $|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\langle H| = (1 \ 0)$
- $|V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\langle V| = (0 \ 1)$
- $\alpha |H\rangle + \beta |V\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ and $\alpha^* \langle H| + \beta^* \langle V| = (\alpha^* \ \beta^*)$

2) In Dirac notation:

$$\begin{aligned} & (\gamma^* \langle H| + \delta^* \langle V|) (\alpha |H\rangle + \beta |V\rangle) \\ &= \gamma^* \alpha \langle H|H\rangle + \gamma^* \beta \langle H|V\rangle + \delta^* \alpha \langle V|H\rangle + \delta^* \beta \langle V|V\rangle \\ &= \gamma^* \alpha + \delta^* \beta \end{aligned}$$

because $\langle H|V\rangle = \langle V|H\rangle = 0$ and $\langle H|H\rangle = \langle V|V\rangle = 1$.

The equivalent vector notation is

$$(\gamma^* \ \delta^*) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \gamma^* \alpha + \delta^* \beta.$$

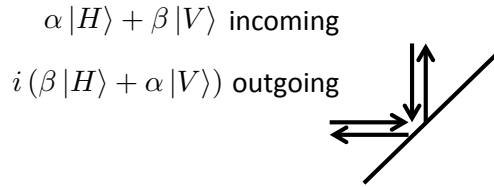
3) We have $R^\top = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and $R^\dagger = R^{\top,*} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$. Thus

$$\begin{aligned} RR^\dagger &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ R^\dagger R &= \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Matrices satisfying $MM^\dagger = M^\dagger M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are called unitary matrices.

Let us compute $R(\alpha |H\rangle + \beta |V\rangle)$ in Dirac notation. By linearity of matrix operations,

$$\begin{aligned} R(\alpha |H\rangle + \beta |V\rangle) &= \alpha R|H\rangle + \beta R|V\rangle \\ &= \alpha i |V\rangle + \beta i |H\rangle \\ &= i(\alpha |V\rangle + \beta |H\rangle). \end{aligned}$$

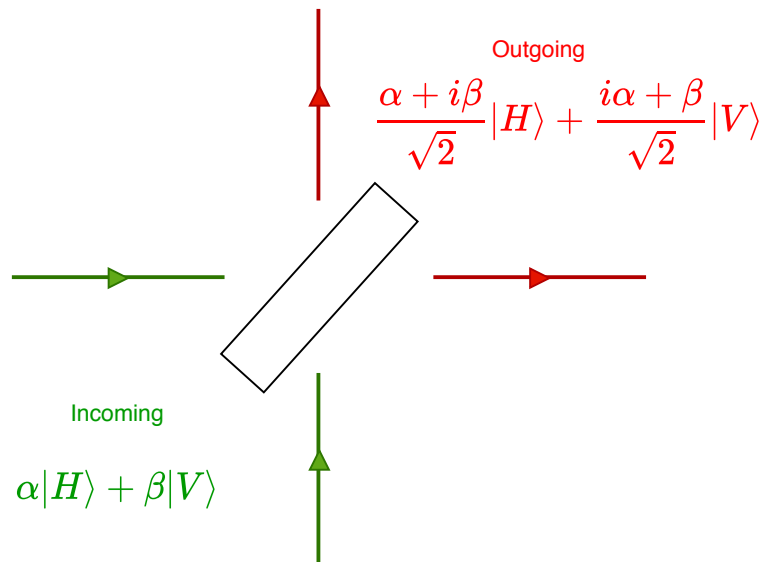


4) We have

$$S |H\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \\ = \frac{1}{\sqrt{2}} (|H\rangle + i|V\rangle),$$

$$S |V\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} \\ = \frac{1}{\sqrt{2}} (i|H\rangle + |V\rangle),$$

$$S(\alpha |H\rangle + \beta |V\rangle) = \alpha S |H\rangle + \beta S |V\rangle \\ = \frac{\alpha}{\sqrt{2}} (|H\rangle + i|V\rangle) + \frac{\beta}{\sqrt{2}} (i|H\rangle + |V\rangle) \\ = \frac{\alpha + i\beta}{\sqrt{2}} |H\rangle + \frac{i\alpha + \beta}{\sqrt{2}} |V\rangle.$$



5) The semi-transparent mirror leaves photons in state $S(\alpha |H\rangle + \beta |V\rangle)$. The state is then measured by the detector D which detects photons in state $|V\rangle$. Therefore, the probability of finding a photon in D is the probability of finding a photon in state $|V\rangle$ given that

photons in state $S(\alpha|H\rangle + \beta|V\rangle)$ are produced. By the measurement postulate (which will be formally introduced in Chapter 3 of the lecture note) we have

$$\text{Prob}(D) = |\langle V|S(\alpha|H\rangle + \beta|V\rangle)|^2.$$

From the previous question we have

$$S(\alpha|H\rangle + \beta|V\rangle) = \frac{\alpha + i\beta}{\sqrt{2}}|H\rangle + \frac{i\alpha + \beta}{\sqrt{2}}|V\rangle,$$

$$\begin{aligned} \langle V|S(\alpha|H\rangle + \beta|V\rangle) &= \frac{\alpha + i\beta}{\sqrt{2}}\langle V|H\rangle + \frac{i\alpha + \beta}{\sqrt{2}}\langle V|V\rangle \\ &= \frac{i\alpha + \beta}{\sqrt{2}}. \end{aligned}$$

So we find

$$\text{Prob}(D) = \left| \frac{i\alpha + \beta}{\sqrt{2}} \right|^2 = \frac{1}{2}|i\alpha + \beta|^2 = \frac{1}{2}(\alpha^2 + \beta^2) = \frac{1}{2}.$$

6) The state after S is

$$S|H\rangle = \frac{1}{\sqrt{2}}(|H\rangle + i|V\rangle)$$

The state after R is

$$\begin{aligned} RS|H\rangle &= \frac{1}{\sqrt{2}}(R|H\rangle + iR|V\rangle) \\ &= \frac{i}{\sqrt{2}}(|V\rangle + i|H\rangle). \end{aligned}$$

The state after the second S is

$$\begin{aligned} SRS|H\rangle &= \frac{i}{\sqrt{2}}(S|V\rangle + iS|H\rangle) \\ &= \frac{i}{\sqrt{2}}\left(\frac{i|H\rangle + |V\rangle}{\sqrt{2}} + i \cdot \frac{|H\rangle + i|V\rangle}{\sqrt{2}}\right) \\ &= -|H\rangle. \end{aligned}$$

Thus

$$\text{Prob}(D_1) = |\langle V|H\rangle|^2 = 0$$

$$\text{Prob}(D_2) = |\langle H|H\rangle|^2 = 1.$$

All photons are detected in D_2 ! For “classical balls” we would expect a split between D_1 and D_2 . For example, if S act as half-half splitters we would expect $\text{Prob}(D_1) = \text{Prob}(D_2) = 1/2$. **The quantum behavior is completely different!**