Homework 4
Quantum Information Processing

Exercise 1 *Properties of Pauli matrices*

We collect useful properties of Pauli matrices. Let $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ a vector formed by the 3 Pauli matrices :

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The identity matrix is denoted $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

a) Show that all 2×2 matrices, A, can be written as a linear combination of I and σ_x, σ_y, σ_z :

$$A = a_0 I + a_1 \sigma_x + a_2 \sigma_y + a_3 \sigma_z.$$

This can also be written as $A = a_0 I + \vec{a} \cdot \vec{\sigma}$ where $\vec{a} \cdot \vec{\sigma}$ is an "inner product" between the "vectors" $\vec{a} = (a_1, a_2, a_3)$ et $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$.

Check also that if $A = A^\dagger$ we have a_0, a_1, a_2, $a_3 \in \mathbb{R}$.

b) Check the following algebraic identities :

$$\begin{aligned} \sigma_x^2 &= \sigma_y^2 = \sigma_z^2 = I \\ \sigma_x \sigma_y &= i \sigma_z \\ \sigma_y \sigma_z &= i \sigma_x \\ \sigma_z \sigma_x &= i \sigma_y \end{aligned}$$

Deduce

$$\begin{aligned} \sigma_x \sigma_y + \sigma_y \sigma_x &= 0 \\ \sigma_y \sigma_z + \sigma_z \sigma_y &= 0 \\ \sigma_z \sigma_x + \sigma_x \sigma_z &= 0 \end{aligned}$$

c) Let $[A, B] = AB - BA$ be the "commutator". Show (you may use preceding results)

$$\begin{aligned} [\sigma_x, \sigma_y] &= 2i\sigma_z \\ [\sigma_y, \sigma_z] &= 2i\sigma_x \\ [\sigma_z, \sigma_x] &= 2i\sigma_y \end{aligned}$$

These relations are called "commutation relations for spin".

d) Compute eigenvalues and eigenvectors of σ_x, σ_y, σ_z. Check that the eigenvalues satisfy $\mathrm{Tr}\,\sigma_x = \mathrm{Tr}\,\sigma_y = \mathrm{Tr}\,\sigma_z = 0$ et $\det\sigma_x = \det\sigma_y = \det\sigma_z = -1$.

e) Dirac notation : set

$$|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix} \text{ et } |\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

Check that

$$\begin{aligned}\sigma_z &= |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow| \\ \sigma_x &= |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow| \\ \sigma_y &= i\,|\downarrow\rangle\langle\uparrow| - i\,|\uparrow\rangle\langle\downarrow|\end{aligned}$$

Exercise 2 *Exponentials of Pauli matrices*

a) We define the exponential of a matrix A by (for $t \in \mathbb{R}$)

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + ...$$

We want to prove the identity :

$$e^{it\vec{n}\cdot\vec{\sigma}} = I\cos t + i\vec{n}\cdot\vec{\sigma}\sin t$$

where $\vec{n}$ is a unit vector and $t \in \mathbb{R}$. Remark that this is a generalization of Euler's identity :

$$e^{i\theta} = \cos\theta + i\sin\theta$$

To show the identity show first that :

$$(\vec{n}\cdot\vec{\sigma})^2 = I$$

Use Taylor expansions of $\cos t$ and $\sin t$ to deduce the wanted identity above.

b) Explicitly write 2×2 matrices (in component/array form) $\exp(it\sigma_x)$; $\exp(it\sigma_y)$; $\exp(it\sigma_z)$ as well as $\exp(it\vec{n}\cdot\vec{\sigma})$.

Exercise 3 *Rotations on the Bloch sphere*

a) Represent the eigenvectors of σ_x, σ_y et σ_z on the Bloch sphere.

b) Calculate explicitly the matrices $\exp\left(-i\frac{\alpha}{2}\sigma_x\right)$, $\exp\left(-i\frac{\alpha}{2}\sigma_y\right)$, $\exp\left(-i\frac{\alpha}{2}\sigma_z\right)$.

c) Consider the qubit $|\psi\rangle = (\cos\frac{\theta}{2})|\uparrow\rangle + e^{i\frac{\pi}{2}}(\sin\frac{\theta}{2})|\downarrow\rangle$. Calculate the action of the matrices $\exp\left(-i\frac{\gamma}{2}\sigma_z\right)$, $\exp\left(-i\frac{\alpha}{2}\sigma_x\right)$, $\exp\left(-i\frac{\beta}{2}\sigma_y\right)$ on this vector. Represent the "trajectory" as a function of α on the Bloch sphere.