Homework 5
Quantum Information Processing

Exercise 1 *Dynamics of Spin 1/2*

We consider a magnetic moment with spin 1/2 whose dynamics is described by a Hamiltonian of the form

$$H = \frac{\hbar\delta}{2}\sigma_z - \frac{\hbar\omega_1}{2}\sigma_x$$

where $\hbar$ is Planck's constant, δ and $\omega_1 \in \mathbb{R}$ and the Pauli matrices $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We recall the formula :

$$\exp\left(i\frac{a}{2}\,\mathbf{n}\cdot\vec{\sigma}\right) = (\cos\frac{a}{2})I + i(\sin\frac{a}{2})\mathbf{n}\cdot\vec{\sigma}$$

with $a \in \mathbb{R}$ et $\mathbf{n} = (n_x, n_y, n_z)$ a unit vector, $\mathbf{n}\cdot\vec{\sigma} = n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$, and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

a) Compute the evolution matrix (operator)

$$U(t,0) = \exp\left(-i\frac{t}{\hbar}H\right)$$

and express it in matrix form, and also in Dirac's notation. We recall the conventional coordinate representation $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

b) Consider the case $\omega_1 << \delta$ and the initial state at $t = 0$, $\frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$.
- Compute a good approximation of the state at time t (hint : take the limit $\omega_1 \to 0$ and δ fixed).
- Represent the trajectory on the Bloch sphere in this limit.
- Is it periodic ? If yes what is the period ?

c) Consider now the case $\delta << \omega_1$ and the initial state at $t = 0$, $|\uparrow\rangle$.
- Compute a good approximation of the final state at time t (hint : take the limit $\delta \to 0$ and ω_1 fixed).
- Represent again the trajectory on the Bloch sphere in this limit.
- Is it periodic ? If yes what is the period ?

Exercise 2 *Creation of entanglement thanks to a magnetic interaction*

We consider two spin $\frac{1}{2}$ with interaction Hamiltonian $\mathcal{H} = \hbar J \sigma_1^z \otimes \sigma_2^z$ (these can be spins of nuclei in a molecule say). The unitary evolution operator of this system is $U = \exp\left(-\frac{it}{\hbar}\mathcal{H}\right)$. Let

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \otimes \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$$

be the initial state of the two spins.

a) Show that the state after time $t = \frac{\pi}{4J}$ is

$$|\psi_t\rangle = \frac{e^{-\frac{i\pi}{4}}}{2}(|\uparrow\uparrow\rangle - i|\uparrow\downarrow\rangle + i|\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle)$$

b) Show that this state is entangled, i.e., it is *impossible* to write it in the form

$$(\alpha|\uparrow\rangle + \beta|\downarrow\rangle) \otimes (\gamma|\uparrow\rangle + \delta|\downarrow\rangle)$$

c) Now we let the state obtained above still evolve for an interval of time $\frac{\pi}{4J}$. Calculate the final state and determine if it is entangled or not.

d) What happens if we let the initial state $|\Psi_0\rangle$ evolve during an interval of time $\frac{\pi}{J}$?

Exercise 3 *The no-cloning theorem*

We want to prove that *non-orthogonal* states cannot be "copied" with the *same* unitary matrix.

In other words let $|\phi_1\rangle$, $|\phi_2\rangle$ in some Hilbert space $\mathcal{H}$ and let $|O\rangle$ be a "blank" state which plays the role of a place-holder for the copy. The theorem states that there does not exist a $U : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ such that,

$$U|\phi_i\rangle \otimes |O\rangle = |\phi_i\rangle \otimes |\phi_i\rangle, \quad i = 1, 2$$

Use the following hints to find two proofs of this fact

a) The first proof only uses only that U is unitary.
Hint : assume a unitary U exists that satisfies the above two equations (for $i = 1, 2$) and find a contradiction.

b) The second proof uses the superposition principle and the linearity of U (it does not really need unitarity).
Hint : consider the action of a linear matrix U that satisfies the above equations on a state which can be writen in two equivalent ways :

$$(|\phi_1\rangle + |\phi_2\rangle) \otimes |O\rangle = |\phi_1\rangle \otimes |O\rangle + |\phi_2\rangle \otimes |O\rangle$$

and find a contradiction.