Homework 9: Solution
Quantum Information Processing

Exercise 1 *Bell states*

1) One has to show that $\langle B_{x,y}|B_{x',y'}\rangle = \delta_{x,x'}\delta_{y,y'}$. We show it explicitly for two cases:

$$\begin{aligned}\langle B_{00}|B_{00}\rangle &= \frac{1}{2}(\langle 00| + \langle 11|)(|00\rangle + |11\rangle)\\ &= \frac{1}{2}(\langle 00|00\rangle + \langle 00|11\rangle + \langle 11|00\rangle + \langle 11|11\rangle).\end{aligned}$$

Now we have

$$\begin{aligned}\langle 00|00\rangle = \langle 0|0\rangle \langle 0|0\rangle = 1, \langle 00|11\rangle = \langle 0|1\rangle \langle 0|1\rangle = 0,\\ \langle 11|00\rangle = \langle 1|0\rangle \langle 1|0\rangle = 0, \langle 11|11\rangle = \langle 1|1\rangle \langle 1|1\rangle = 1.\end{aligned}$$

Thus we get that $\langle B_{00}|B_{00}\rangle = \frac{1}{2}(1+0+0+1) = 1$. Now let us consider

$$\begin{aligned}\langle B_{00}|B_{01}\rangle &= \frac{1}{2}(\langle 00| + \langle 11|)(|01\rangle + |10\rangle)\\ &= \frac{1}{2}(\langle 00|01\rangle + \langle 00|10\rangle + \langle 11|01\rangle + \langle 11|10\rangle)\\ &= \frac{1}{2}(0+0+0+0) = 0.\end{aligned}$$

2) The proof is by contradiction. Suppose there exists a_1, b_1 and a_2, b_2 such that

$$|B_{00}\rangle = (a_1 |0\rangle + b_1 |1\rangle) \otimes (a_2 |0\rangle + b_2 |1\rangle).$$

Then we have

$$\frac{1}{2}(|00\rangle + |11\rangle) = a_1 a_2 |00\rangle + a_1 b_2 |01\rangle + b_1 a_2 |10\rangle + a_2 b_2 |11\rangle .$$

Comparing the coefficients of the orthornormal basis, one has

$$\frac{1}{2} = a_1 a_2,\ \frac{1}{2} = b_1 b_2,\ a_1 b_2 = 0,\ b_1 a_2 = 0.$$

The third equality indicates that either $a_1 = 0$ or $b_2 = 0$ (or both). If $a_1 = 0$ we get a contradiction with the first equation. If on the other hand $b_2 = 0$, we get a contradiction with the second one. Therefore, there does not exist $|\psi_1\rangle$ and $|\psi_2\rangle$ such that $|B_{00}\rangle$ can be written as $|\psi_1\rangle \otimes |\psi_2\rangle$. Therefore, B_{00} is entangled.

3) We have

$$\begin{aligned}|\gamma\rangle \otimes |\gamma\rangle &= (\cos(\gamma)|0\rangle + \sin(\gamma)|1\rangle) \otimes (\cos(\gamma)|0\rangle + \sin(\gamma)|1\rangle) \\ &= \cos^2(\gamma)|00\rangle + \cos(\gamma)\sin(\gamma)|01\rangle + \sin(\gamma)\cos(\gamma)|10\rangle + \sin^2(\gamma)|11\rangle .\end{aligned}$$

Similarly, we have

$$|\gamma_\perp\rangle \otimes |\gamma_\perp\rangle = = \sin^2(\gamma)|00\rangle - \cos(\gamma)\sin(\gamma)|01\rangle - \sin(\gamma)\cos(\gamma)|10\rangle + \cos^2(\gamma)|11\rangle .$$

Combining the two terms, we find that

$$\begin{aligned}|\gamma\rangle \otimes |\gamma\rangle + |\gamma_\perp\rangle \otimes |\gamma_\perp\rangle &= (\cos^2(\gamma) + \sin^2(\gamma))|00\rangle + (\sin^2(\gamma) + \cos^2(\gamma))|11\rangle \\ &= |00\rangle + |11\rangle\end{aligned}$$

and thus

$$\frac{1}{\sqrt{2}}(|\gamma\rangle \otimes |\gamma\rangle + |\gamma_\perp\rangle \otimes |\gamma_\perp\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |B_{00}\rangle .$$

4) From the rule of the tensor product

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a\begin{pmatrix} c \\ d \end{pmatrix} \\ \\ b\begin{pmatrix} c \\ d \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix},$$

we obtain the basis states as

$$|0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad |0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$|1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, we have

$$|B_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad |B_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix},$$

$$|B_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \qquad |B_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

Exercise 2 *Entanglement by unitary operations*

1) By definition of the tensor product:

$$(H \otimes I)\left|x\right\rangle \otimes \left|y\right\rangle = H\left|x\right\rangle \otimes I\left|y\right\rangle = H\left|x\right\rangle \otimes \left|y\right\rangle.$$

Also, one can use that $H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ to show that always

$$H\left|x\right\rangle = \frac{1}{\sqrt{2}}(\left|0\right\rangle + (-1)^x\left|1\right\rangle).$$

Thus,

$$(H \otimes I)\left|x\right\rangle \otimes \left|y\right\rangle = \frac{1}{\sqrt{2}}(\left|0\right\rangle \otimes \left|y\right\rangle + (-1)^x\left|1\right\rangle \otimes \left|y\right\rangle).$$

Now we apply CNOT. By linearity, we can apply it to each term separately. Thus,

$$\begin{aligned}(\text{CNOT})(H \otimes I)\left|x\right\rangle \otimes \left|y\right\rangle &= \frac{1}{\sqrt{2}}((CNOT)\left|0\right\rangle \otimes \left|y\right\rangle + (-1)^x(\text{CNOT})\left|1\right\rangle \otimes \left|y\right\rangle)\\ &= \frac{1}{\sqrt{2}}(\left|0\right\rangle \otimes \left|y\right\rangle + (-1)^x\left|1\right\rangle \otimes \left|y \oplus 1\right\rangle)\\ &= \left|B_{xy}\right\rangle.\end{aligned}$$

2) Let us first start with $H \otimes I$. We use the rule

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} \otimes \begin{pmatrix}e & f\\ g & h\end{pmatrix} = \begin{pmatrix}ae & af & be & bf\\ ag & ah & bg & bh\\ ce & cf & de & df\\ cg & ch & dg & dh\end{pmatrix},$$

Thus we have

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\end{pmatrix}.$$

For CNOT, we use the definition:

$$(\text{CNOT})\left|x\right\rangle \otimes \left|y\right\rangle = \left|x\right\rangle \otimes \left|y \oplus x\right\rangle,$$

which implies that the matrix elements are

$$\left\langle x'y'\right|\text{CNOT}\left|xy\right\rangle = \left\langle x', y' | x, y \otimes x\right\rangle = \left\langle x' | x\right\rangle\left\langle y' | y \oplus x\right\rangle = \delta_{xx'}\delta_{y\oplus x, y'}.$$

We obtain the following table with columns xy and rows $x'y'$:

	00	01	10	11
00	1	0	0	0
01	0	1	0	0
10	0	0	0	1
11	0	0	1	0

For the matrix product (CNOT)$(H \otimes I)$, we find that

$$
\begin{aligned}
(\text{CNOT})H \otimes I &= \frac{1}{\sqrt{2}} \begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ X & -X \end{pmatrix},
\end{aligned}
$$

where $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Thus,

$$
(CNOT)(H \otimes I) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}.
$$

One can check that for example $|B_{00}\rangle = (\text{CNOT})(H \otimes I)\,|0\rangle \otimes |0\rangle$. Finally to check the unitarity, we have to check that $UU^\dagger = U^\dagger U = I$ for $U = H \otimes I$, CNOT and (CNOT)$(H \otimes I)$. We leave this to the reader.

3) Let $U = (\text{CNOT})(H \otimes I)$. We have

$$
|B_{xy}\rangle = U\,|x\rangle \otimes |y\rangle\,, \langle B_{x'y'}| = \langle x'| \otimes \langle y'|\, U^\dagger.
$$

Thus,

$$
\begin{aligned}
\langle B_{x'y'}|B_{xy}\rangle &= \langle x'| \otimes \langle y'|\, U^\dagger U\,|x\rangle \otimes |y\rangle \\
&= \langle x'| \otimes \langle y'|\, I\,|x\rangle \otimes |y\rangle \\
&= \langle x'|x\rangle\,\langle y'|y\rangle = \delta_{xx'}\delta_{yy'}.
\end{aligned}
$$

Exercise 3 *Tsirelson bound*

1) Since the eigenvalues of A are ± 1, those of A^2 are both $+1$. The eigenvectors are the same. Thus we have $A^2 = |\alpha\rangle\langle\alpha| + |\alpha_\perp\rangle\langle\alpha_\perp| = I_A$ and similarly for the other matrices. The same result can also be obtained by expanding the product A^2 in Dirac notation or in matrix component form...

   Now expanding $\mathcal{B}^2$ we get the squared terms of the type $(A \otimes B)^2 = (A \otimes B)(A \otimes B) = A^2 \otimes B^2 = I_A \otimes I_B$. This will yield the term $4 I_A \otimes I_B$.

   Then for the cross terms using $(M_1 \otimes M_2)(N_1 \otimes N_2) = M_1 N_1 \otimes M_2 N_2$ we end up with 4 contributions that dont cancel each other and can be rearranged into the commutator term $[A, A'] \otimes [B, B']$.

2) From the identity we have

$$
\begin{aligned}
\langle\Psi|\mathcal{B}^2|\Psi\rangle &= 4 - \langle\Psi|[A, A'] \otimes [B, B']|\Psi\rangle \\
&\leq 4 + |\langle\Psi|[A, A'] \otimes [B, B']|\Psi\rangle|
\end{aligned}
$$

Now we must prove that the last term on the right-hand-side is less or equal to 4. Using the hint and Cauchy-Schwarz

$$|\langle\Psi|[A, A'] \otimes [B, B']|\Psi\rangle| \leq \|[A, A'] \otimes I_B|\Psi\rangle\|\|I_A \otimes [B, B']\Psi\rangle\|$$

To estimate each norm on the right-hand-side we first use the triangle inequality

$$\|[A, A'] \otimes I_B|\Psi\rangle\| \leq \|AA' \otimes I_B|\Psi\rangle\| + \|A'A \otimes I_B|\Psi\rangle\|$$

and then

$$\begin{aligned}
\|AA' \otimes I_B|\Psi\rangle\|^2 &= \langle\Psi|(AA' \otimes I_B)^\dagger(AA' \otimes I_B)|\Psi\rangle \\
&= \langle\Psi|(A'A \otimes I_B)(AA' \otimes I_B)|\Psi\rangle \\
&= \langle\Psi|A'A^2A' \otimes I_B^2|\Psi\rangle \\
&= \langle\Psi|A'^2 \otimes I_B|\Psi\rangle \\
&= \langle\Psi|I_A \otimes I_B|\Psi\rangle = 1
\end{aligned} \tag{1}$$

From which we deduce that the right-hand-side of the last inequality above equals $1+1 = 2$. Thus

$$\|[A, A'] \otimes I_B|\Psi\rangle\| \leq 2$$

and

$$|\langle\Psi|[A, A'] \otimes [B, B']|\Psi\rangle| \leq 4$$

which completes the proof.

3) This last result can be justified from Cauchy-Schwarz:

$$\langle\Psi|\mathcal{B}|\Psi\rangle^2 \leq \||\Psi\rangle\|^2\|\mathcal{B}|\Psi\rangle\|^2 = \langle\Psi|\mathcal{B}^2|\Psi\rangle$$

Since we proved before that teh r.h.s is less or equal to 8 we get Tsirelson's bound for any state

$$\langle\Psi|\mathcal{B}|\Psi\rangle \leq 2\sqrt{2}$$

Note that the Bell state saturates the bound.