Homework 13

Exercise 1*. Let $(X_n,\, n \geq 1)$ be a sequence of i.i.d. random variables such that $\mathbb{P}(\{X_1 = +1\}) = \mathbb{P}(\{X_1 = -1\}) = \frac{1}{2}$. Let also $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ for $n \geq 1$ and let $(H_n,\, n \in \mathbb{N})$ be a predictable process with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\exists K_n > 0$ with $|H_n(\omega)| \leq K_n$ for all $\omega \in \Omega$. Let finally

$$G_0 = 0 \quad \text{and} \quad G_n = \sum_{j=1}^{n} H_j X_j, \quad n \geq 1.$$

From the course, we know that the process G is a martingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$.

a) Under the assumptions made, is it possible that $\mathbb{E}(H_j X_j) > 0$ for some j? Explain!

b) Find the unique predictable and increasing process $(A_n,\, n \in \mathbb{N})$ such that the process $(G_n^2 - A_n,\, n \in \mathbb{N})$ is also a martingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$.

From now on, consider the particular case where $H_n(\omega) \in \{-1, +1\}$ for every $n \in \mathbb{N}$ and $\omega \in \Omega$.

c) Compute the process A in this particular case.

d) Let $a \geq 1$ be an integer and let $T = \inf\{n \geq 1 \,:\, |G_n| \geq a\}$. Compute $\mathbb{E}(T)$ [no full justification required here].

Exercise 2. Let $0 < p < 1$ and $x > 0$ be fixed real numbers and $(X_n,\, n \in \mathbb{N})$ be the process defined recursively as

$$X_0 = x, \quad X_{n+1} = \begin{cases} X_n^2 + 1 & \text{with probability } p \\ X_n/2 & \text{with probability } 1 - p \end{cases} \quad \text{for } n \in \mathbb{N}$$

a) What *minimal* condition on $0 < p < 1$ guarantees that the process X is a submartingale (with respect to its natural filtration)? Justify your answer.

Hint: The inequality $a^2 + b^2 \geq 2ab$ may be useful here.

b) For the values of p respecting the condition found in part a), derive a lower bound on $\mathbb{E}(X_n)$.

Hint: Proceed recursively.

c) Does there exist a value of $0 < p < 1$ such that the process X is a martingale? a supermartingale? Again, justify your answer.

Exercise 3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Let $U \sim \mathcal{U}([-1,+1])$ be a random variable independent of $\mathcal{G}$ and M be a positive, integrable and $\mathcal{G}$-measurable random variable.

a) Compute the function $\psi : \mathbb{R}_+ \to \mathbb{R}$ satisfying

$$\psi(M) = \mathbb{E}(|M+U| \mid \mathcal{G})$$

Let now $(U_n,\, n \geq 1)$ be a sequence of i.i.d.$\sim \mathcal{U}([-1,+1])$ random variables, all defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(U_1, \ldots, U_n)$, $n \geq 1$. Let finally $(M_n,\, n \geq 1)$ be the process defined recursively as

$$M_0 = 0, \quad M_{n+1} = |M_n + U_{n+1}|, \quad n \in \mathbb{N}$$

b) Show that the process $(M_n,\, n \in \mathbb{N})$ is a submartingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$.

c) Compute the unique predictable and increasing process $(A_n,\, n \in \mathbb{N})$ such that the process $(M_n - A_n,\, n \in \mathbb{N})$ is a martingale with respect to $(\mathcal{F}_n, ,\, n \in \mathbb{N})$.

d) Is it true that the process $(M_n^2,\, n \in \mathbb{N})$ is also a submartingale with respect to $(\mathcal{F}_n, ,\, n \in \mathbb{N})$? Justify your answer.

e) Determine the value of $c > 0$ such that the process $(N_n = M_n^2 - cn, n \in \mathbb{N})$ is a martingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$.

f) Does there exist a random variable M_∞ such that $M_n \underset{n\to\infty}{\to} M_\infty$ almost surely? (Again, no formal justification required here; an intuitive argument will do.)