Solutions to Homework 14

Exercise 1*. a) Observe first that for every value of $0 < p < 1$, we have $0 < M_{n+1} < 1$ whenever $0 < M_n < 1$ (and $M_0 = x \in]0,1[$ by assumption), so that it makes sense to consider both M_n and $1 - M_n$ as probabilities at every step. Let us then compute:

$$\mathbb{E}(M_{n+1} \,|\, \mathcal{F}_n) = pM_n\,(1 - M_n) + ((1-p) + pM_n)\,M_n = M_n$$

for every $0 < p < 1$!

b) Again, let us compute:

$$\mathbb{E}(M_{n+1}\,(1 - M_{n+1}) \,|\, \mathcal{F}_n) = pM_n\,(1 - pM_n)\,(1 - M_n) + ((1-p) + pM_n)\,(1 - (1-p) - pM_n)\,M_n$$
$$= pM_n\,(1 - pM_n)\,(1 - M_n) + ((1-p) + pM_n)\,p(1 - M_n)\,M_n = p(2-p)\,M_n\,(1 - M_n)$$

c) Therefore, we obtain by induction:

$$\mathbb{E}(M_n\,(1 - M_n)) = (p(2-p))^n\,x(1-x)$$

which converges to 0 as n gets large (as $p(2-p) < 1$ for every $0 < p < 1$).

d) Because the martingale M is bounded, the answer is yes to all three questions.

e) The answer obtained in c) suggests that M_n converges either to 0 or 1 as $n \to \infty$, which turns out to be the case. Moreover, as seen in class:

$$\mathbb{P}(\{M_\infty = 1\}) = \mathbb{E}(M_\infty) = \mathbb{E}(M_0) = x, \quad \text{so} \quad \mathbb{P}(\{M_\infty = 0\}) = 1 - x$$

Exercise 2.

Preliminary question. This question just requires a direct application of the generalized version of Hoeffding's inequality.

Part I.

a) As $0 < p < 1/2$, S is a supermartingale. Also, $\lambda^x = e^{x \log(\lambda)}$ is a convex function $\forall \lambda > 0$, but is increasing for $\lambda \geq 1$ and decreasing for $\lambda \leq 1$. Therefore, applying Jensen's inequality gives

$$\mathbb{E}(\lambda^{S_{n+1}} \,|\, \mathcal{F}_n) \geq \lambda^{\mathbb{E}(S_{n+1} \,|\, \mathcal{F}_n)} \geq \lambda^{S_n}$$

only when $\lambda \leq 1$.

b) We have

$$\mathbb{E}(\lambda^{S_{n+1}} \,|\, \mathcal{F}_n) = \lambda^{S_n}\,\mathbb{E}(\lambda^{X_{n+1}}) = \lambda^{S_n}\,(\lambda p + \frac{1}{\lambda}(1-p))$$

which says that Y is a submartingale if and only if $E = \lambda p + \frac{1}{\lambda}(1-p) \geq 1$. Solving this (quadratic) inequation, we obtain the condition: $\lambda \in]-\infty, 1] \cup [\frac{1-p}{p}, +\infty[$. For the martingale condition to hold ($E = 1$), it must be that either $\lambda = 1$ (trivial case) or $\lambda = \frac{1-p}{p}$. Finally, Y is a supermartingale ($E \leq 1$) if and only if $\lambda \in [1, \frac{1-p}{p}]$.

c) By independence, we have $\mathbb{E}(|Y_n|) = \mathbb{E}(Y_n) = \prod_{j=1}^n \mathbb{E}(\lambda^{X_j}) = (\lambda p + \frac{1}{\lambda}(1-p))^n$ and

$$\mathbb{E}(Y_n^2) = \prod_{j=1}^n \mathbb{E}(\lambda^{2X_j}) = (\lambda^2 p + \frac{1}{\lambda^2}(1-p))^n$$

d) Using the analysis done in question b), we obtain that

$$\sup_{n\in\mathbb{N}} \mathbb{E}(|Y_n|) < +\infty \quad \text{if and only if} \quad \lambda p + \frac{1}{\lambda}(1-p) \leq 1 \quad \text{if and only if} \quad \lambda \in [1, \frac{1-p}{p}]$$

Likewise,

$$\sup_{n\in\mathbb{N}} \mathbb{E}(Y_n^2) < +\infty \quad \text{if and only if} \quad \lambda^2 p + \frac{1}{\lambda^2}(1-p) \leq 1 \quad \text{if and only if} \quad \lambda \in [1, \sqrt{\frac{1-p}{p}}]$$

as we simply need to replace λ by λ^2 here.

e) Numerically (see the code), we see clearly that $Y_n \equiv 1$ when $\lambda = 1$ and that $Y_n \underset{n\to\infty}{\to} 0$ a.s. for all $\lambda > 1$. Theoretically, this is due to the fact that $S_n \underset{n\to\infty}{\to} -\infty$ a.s. (one can use Hoeffding's inequality to justify this fact), so that $Y_n = \lambda^{S_n}$ converges a.s. to 0 when $\lambda > 1$ (even though Y is a submartingale for $\lambda > \frac{1-p}{p}$).

The martingale convergence theorem itself only allows to conclude that Y converges a.s. when $\lambda \in [1, \frac{1-p}{p}]$, i.e., when the first condition in question d) is satisfied.

f) For the process Y to converge also in L^2 towards its limit, we need the second condition in part d) to hold, i.e., $\lambda \in [1, \sqrt{\frac{1-p}{p}}]$ (indeed, if this condition is not met, then $\mathbb{E}(Y_n^2)$ diverges, so no L^2 convergence can take place).

g) The only value of λ for which the equality $\mathbb{E}(Y_\infty|\mathcal{F}_n) = Y_n$ is satisfied for all $n \in \mathbb{N}$ is the trivial case $\lambda = 1$. For all the other values of $\lambda \in [1, \sqrt{\frac{1-p}{p}}]$, Y is a supermartingale, which gives $\mathbb{E}(Y_\infty|\mathcal{F}_n) \leq Y_n$ for all $n \in \mathbb{N}$. But in this case, we already know that $Y_\infty = 0$, so what the above inequality is actually saying is that $0 \leq Y_n$, i.e., not much...

Part II. We consider the case $\lambda = \frac{1-p}{p}$ (> 1).

a) Method: observe first that $T_a = \inf\{n \in \mathbb{N} : S_n \geq a \quad \text{or} \quad S_n \leq -a\}$. Run then $M = 1'000$ times the process S for $N = 1'000$ time steps; count the number of times m the barrier a is reached before $-a$ (see the code for the implementation). The probability $P = \mathbb{P}(\{Y_{T_a} = \lambda^a\})$ is then approximately given by m/M (there are of course fluctuations here).

b) It is true that $\mathbb{E}(Y_{T_a}) = \mathbb{E}(Y_0)$. The third version of the optional stopping theorem allows to say this, as the martingale Y makes bounded jumps and is contained in the interval $[\lambda^{-a}, \lambda^a]$ before the stopping time T_a.

c) As the martingale convergence theorem applies, we obtain

$$1 = \mathbb{E}(Y_0) = \mathbb{E}(Y_{T_a}) = \lambda^a \, P + \lambda^{-a} \, (1 - P)$$

so

$$P = \frac{1 - \lambda^{-a}}{\lambda^a - \lambda^{-a}} = \frac{\lambda^a - 1}{\lambda^{2a} - 1} = \frac{1}{\lambda^a + 1}$$

This probability decays exponentially in a.

d) Method: Run again $M = 1'000$ times the process S for $N = 1'000$ time steps; count the number of times m the barrier a is reached at all (see the code for the implementation). The probability $P' = \mathbb{P}(\{Y_{T'_a} = \lambda^a\})$ is then approximately given by m/M (there are of course also fluctuations here).

e) It is also true that $\mathbb{E}(Y_{T'_a}) = \mathbb{E}(Y_0)$! The third version of the optional stopping theorem allows to say this, as the martingale Y makes bounded jumps and is contained in the interval $[0, \lambda^a]$ before the stopping time T'_a (even though the process S itself is unbounded).

f) As the martingale convergence theorem applies, we obtain

$$1 = \mathbb{E}(Y_0) = \mathbb{E}(Y_{T'_a}) = \lambda^a \, P' + 0 \, (1 - P')$$

[Comment: if level a is not reached, this is saying that the process S goes to $-\infty$ and never comes back to level a; of course, this can only happen with positive probability when S is a (strict) supermartingale with a negative drift.] Therefore,

$$P' = \frac{1}{\lambda^a}$$

which is strictly greater than P, but decays also exponentially in a.