Homework 12

Exercise 1. Let $(M_n,\, n \in \mathbb{N})$ be a submartingale with respect to a filtration $(\mathcal{F}_n,\, n \in \mathbb{N})$ and $\varphi : \mathbb{R} \to \mathbb{R}$ be a Borel-measurable and convex function such that $\mathbb{E}(|\varphi(M_n)|) < +\infty$, $\forall n \in \mathbb{N}$.

a) What additional property of φ ensures that the process $(\varphi(M_n),\, n \in \mathbb{N})$ is also a submartingale?

b) In particular, which of the following two processes is ensured to be a submartingale: $(M_n^2,\, n \in \mathbb{N})$ and/or $(\exp(M_n),\, n \in \mathbb{N})$?

Let $(X_n,\, n \geq 1)$ be a sequence of i.i.d. random variables such $\mathbb{P}(\{X_1 = +1\}) = \mathbb{P}(\{X_1 = -1\}) = \frac{1}{2}$; let $S_0 = 0$ and $S_n = X_1 + \ldots + X_n$ for $n \geq 1$; finally, let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ for $n \geq 1$.

c) For which value of $c > 0$ is the process $(S_n^2 - cn,\, n \in \mathbb{N})$ is a martingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$?

d) For which value of $c > 0$ is the process $\left(\frac{\exp(S_n)}{c^n},\, n \in \mathbb{N}\right)$ a martingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$?

Assume now that $\mathbb{P}(\{X_1 = +1\}) = p = 1 - \mathbb{P}(\{X_1 = -1\})$ for some $0 < p < 1$ with $p \neq \frac{1}{2}$.

e) Does there exist a number $c > 0$ such that the process $(S_n^2 - cn,\, n \in \mathbb{N})$ is a martingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$? If yes, compute the value of c; otherwise, justify why it is not the case.

f) Does there exist a number $c > 0$ such that the process $\left(\frac{\exp(S_n)}{c^n},\, n \in \mathbb{N}\right)$ is a martingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$? If yes, compute the value of c; otherwise, justify why it is not the case.

Exercise 2*. A bag contains red and blue balls with initially r red and b blue where $rb > 0$. A ball is drawn from the bag, its colour noted, and the it is returned to the bag together with a new ball of the same color. Let R_n be the number of red balls after n such operations.

a) Show that $Y_n = \frac{R_n}{n+r+b}$ is a martingale with respect to the natural filtration $(\mathcal{F}_n,\, n \in \mathbb{N})$.

b) Define a stopping time

$$T = \inf\{n \geq 1 \colon \text{there are } m \text{ red balls or there are } m \text{ blue balls in the bag}\},$$

for some $m > \max\{r, b\}$. What is the expectation of Y_T?

Exercise 3. (If one cannot win on a game, then it is a martingale)
Let $(\mathcal{F}_n,\, n \in \mathbb{N})$ be a filtration and $(M_n,\, n \in \mathbb{N})$ be a process adapted to $(\mathcal{F}_n,\, n \in \mathbb{N})$ such that $\mathbb{E}(|M_n|) < \infty$, for all $n \in \mathbb{N}$.

Show that if for any predictable process $(H_n,\, n \in \mathbb{N})$ such that H_n is a bounded random variable $\forall n \in \mathbb{N}$, we have

$$\mathbb{E}((H \cdot M)_N) = 0, \quad \forall N \in \mathbb{N},$$

then $(M_n,\, n \in \mathbb{N})$ is a martingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$.

Exercise 4. Let $(X_n,\, n \geq 1)$ be a family of independent square-integrable random variables such that $\mathbb{E}(X_n) = 0$ for all $n \geq 1$. Let $M_0 = 0$, $M_n = X_1 + \ldots + X_n$, $n \geq 1$.

The process $(M_n,\, n \in \mathbb{N})$ is a martingale, but it is also a process with independent increments. Show that $(M_n^2 - \mathbb{E}(M_n^2),\, n \in \mathbb{N})$ is also a martingale (hence the process A in the Doob decomposition of the submartingale $(M_n^2,\, n \in \mathbb{N})$ is a deterministic process in this case).