Homework 8

Exercise 1*. a) Let $(X_n, n \geq 1)$ be a sequence of bounded i.i.d. random variables such that $\mathbb{E}(X_1) = 0$ and $\mathrm{Var}(X_1) = 1$, and let $S_n = X_1 + \ldots + X_n$ for $n \geq 1$. Show that the event

$$A = \left\{ \frac{S_n}{n} \text{ converges} \right\}$$

belongs to the tail σ-field $\mathcal{T} = \cap_{n \geq 1} \sigma(X_n, X_{n+1}, \ldots)$ (implying that $\mathbb{P}(A) \in \{0, 1\}$ by Kolomgorov's 0-1 law; but the law of large numbers tells you more in this case, namely that $\mathbb{P}(A) = 1$.).

b) Assume now that $(X_n, n \geq 1)$ is a sequence of bounded, *uncorrelated* and identically distributed random variables such that $\mathbb{E}(X_1) = 0$ and $\mathrm{Var}(X_1) = 1$. The answer to Exercise 3.b) in homework 6 tells you what happens to $\mathbb{P}(A)$ in this case.

That said, under this more general assumption, Kolmogorov's 0-1 law may not necessarily hold. Prove it by exhibiting a sequence of random variables $(X_n, n \geq 1)$ satisfying these assumptions and an event $B \in \mathcal{T}$ such that $0 < \mathbb{P}(B) < 1$.

Exercise 2. Someone proposes you to play the following game: start with an initial amount of $S_0 > 0$ francs, of your choice. Then toss a coin: if it falls on heads, you win $S_0/2$ francs; while if it falls on tails, you lose $S_0/2$ francs. Call S_1 your amount after this first coin toss. Then the game goes on, so that your amount after coin toss number $n \geq 1$ is given by

$$S_n = \begin{cases} S_{n-1} + \frac{S_{n-1}}{2} & \text{if coin number } n \text{ falls on heads} \\ \\ S_{n-1} - \frac{S_{n-1}}{2} & \text{if coin number } n \text{ falls on tails} \end{cases}$$

We assume moreover that the coin tosses are independent and fair, i.e., with probability 1/2 to fall on each side. Nevertheless, you should *not* agree to play such a game: explain why!

Hints:

First, to ease the notation, define $X_n = +1$ if coin n falls on heads and $X_n = -1$ if coin n falls on tails. That way, the above recursive relation may be rewritten as $S_n = S_{n-1}\,(1 + \frac{X_n}{2})$ for $n \geq 1$.

a) Compute recursively $\mathbb{E}(S_n)$; if it were only for expectation, you could still consider playing such a game, but...

b) Define now $Y_n = \log(S_n/S_0)$, and use the central limit theorem to approximate $\mathbb{P}(\{Y_n > t\})$ for a fixed value of $t \in \mathbb{R}$ and a relatively large value of n. Argue from there why it is definitely not a good idea to play such a game! (computing for example an approximate value of $\mathbb{P}(\{S_{100} > S_0/10\})$)

Exercise 3. Let $\lambda > 0$ be fixed. For a given $n \geq \lceil \lambda \rceil$, let $X_1^{(n)}, \ldots, X_n^{(n)}$ be i.i.d. Bernoulli(λ/n) random variables and let $S_n = X_1^{(n)} + \ldots + X_n^{(n)}$.

a) Compute $\mathbb{E}(S_n)$ and $\mathrm{Var}(S_n)$ for a fixed value of $n \geq \lceil \lambda \rceil$.

b) Deduce the value of $\mu = \lim_{n\to\infty} \mathbb{E}(S_n)$ and $\sigma^2 = \lim_{n\to\infty} \mathrm{Var}(S_n)$.

c) Compute the limiting distribution of S_n (as $n \to \infty$).

Hint: Use characteristic functions. You might also have a look at tables of characteristic functions of some well known distributions in order to solve this exercise.

For a given $n \geq 1$, let now $Y_1^{(n)}, \dots, Y_n^{(n)}$ be i.i.d. Bernoulli$(1/n)$ random variables and let

$$T_n = Y_1^{(n)} + \ldots + Y_{\lceil \lambda n \rceil}^{(n)}$$

where $\lambda > 0$ is the same as above.

d) Compute the limiting distribution of T_n (as $n \to \infty$).

e) Is it also the case that either S_n or T_n converge almost surely or in probability towards a limit? Justify your answer!