Homework 9

Exercise 1. Let $(m_k,\, k \geq 0)$ be the sequence of moments of a generic random variable X.

a) Let $\ell \geq k \geq 0$. Show that if $\mathbb{E}(|X|^\ell) < +\infty$ (i.e., if m_ℓ is well-defined and finite), then it also holds that $\mathbb{E}(|X|^k) < +\infty$.

Note: The reciprocal statement is that if $\ell \geq k \geq 0$ and $\mathbb{E}(|X|^k) = +\infty$, then $\mathbb{E}(|X|^\ell) = +\infty$.

b) Show that the growth of the odd moments is controlled by the growth of the even moments.

c) Show that if X is bounded, then Carleman's condition is satisfied:

$$\sum_{k \geq 1} m_{2k}^{-\frac{1}{2k}} = +\infty$$

Exercise 2.* a) Let $X \sim \mathcal{N}(0,1)$. Compute all the moments of the random variable $Y = \exp(X)$.

b) Let W be the discrete random variable such that

$$\mathbb{P}(\{W = j\}) = C\, \exp(-j^2/2), \quad j \in \mathbb{Z}$$

where $C = 1/\sum_{j \in \mathbb{Z}} \exp(-j^2/2)$. Compute all the moments of the random variable $Z = \exp(W)$.

c) What can you conclude from parts a) and b)?

Exercise 3. Let X be a $\mathcal{N}(0, \sigma^2)$ random variable, with $\sigma > 0$.

a) Using integration by parts, show that for any continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that there exists $C > 0$ and $n \geq 1$ with $|f(x)|, |f'(x)| \leq C\,(1 + x^2)^n$ for every $x \in \mathbb{R}$, we have

$$\mathbb{E}(X \cdot f(X)) = \sigma^2\, \mathbb{E}(f'(X))$$

Note: The above condition is needed to ensure that both expectations are finite.

b) Use part a) to compute $\mathbb{E}(X^k)$ for $k \in \mathbb{N}$.

Let now $m \geq 1$ and $Y = X^m$.

c) For which values of $m \geq 1$ and $\sigma > 0$ does it hold that the distribution of Y is entirely determined by its moments? (*Hint:* use Stirling's approximation: $k! \simeq k^k\, e^{-k}$).