Homework 4

Exercise 1. Let $\lambda > 0$ and $X \sim \mathcal{E}(\lambda)$, and let us define $Y = X^a$, where $a \in \mathbb{R}$.

a) For what values of $a \in \mathbb{R}$ does it hold that $\mathbb{E}(Y) < +\infty$?

b) For what values of $a \in \mathbb{R}$ does it hold that $\mathbb{E}(Y^2) < +\infty$?

c) For what values of $a \in \mathbb{R}$ is $\text{Var}(Y)$:

c1) well-defined and finite? c2) well-defined but infinite? c3) ill-defined?

d) Compute $\mathbb{E}(Y)$ and $\text{Var}(Y)$ for the values of $a \in \mathbb{Z}$ such that these quantities are well-defined.

Hint: Use integration by parts, recursively.

Exercise 2. Let X be a random variable that is symmetrically distributed (i.e. $X \sim -X$) and square-integrable with $\text{Var}(X) = 1$. Let also $Y = 1_{\{X \geq 0\}}$.

a) Show that for any distribution of the random variable X, $\text{Cov}(X, Y) \geq 0$.

b) Using the inequality $\text{Cov}(X, Y) \leq \sqrt{\text{Var}(X)}\,\sqrt{\text{Var}(Y)}$ (whose proof is to come in the sequel of the course), find the least value $C > 0$ such that $\text{Cov}(X, Y) \leq C$ for every distribution of X.

c) Compute $\text{Cov}(X, Y)$ for $X \sim \mathcal{N}(0, 1)$.

d) Is it possible to find a distribution for X such that $\text{Cov}(X, Y) = C$? If not, is it possible to find a sequence of random variables $(X_n,\, n \geq 1)$ with varying distributions (all respecting the above constraints) and $Y_n = 1_{\{X_n \geq 0\}}$, such that $\text{Cov}(X_n, Y_n) \underset{n\to\infty}{\to} C$?

e) Is it possible to find a distribution for X such that $\text{Cov}(X, Y) = 0$? If not, is it possible to find a sequence of random variables $(X_n,\, n \geq 1)$ with varying distributions (all respecting the above constraints) and $Y_n = 1_{\{X_n \geq 0\}}$, such that $\text{Cov}(X_n, Y_n) \underset{n\to\infty}{\to} 0$?

Exercise 3. For a generic *non-negative* random variable X defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, it holds that (the exchange of expectation and integral sign is valid here):

$$\mathbb{E}(X) = \mathbb{E}\left(\int_0^X dt\right) = \mathbb{E}\left(\int_0^{+\infty} 1_{\{X \geq t\}}\, dt\right) = \int_0^{+\infty} \mathbb{E}(1_{\{X \geq t\}})\, dt = \int_0^{+\infty} \mathbb{P}(\{X \geq t\})\, dt$$

a) Use this formula to compute $\mathbb{E}(X)$ for $X \sim \mathcal{E}(\lambda)$.

b) Particularize the above formula for $\mathbb{E}(X)$ to the case where X takes values in $\mathbb{N}$ only.

c) Use this new formula to compute $\mathbb{E}(X)$ for $X \sim \text{Bern}(p)$ and $X \sim \text{Geom}(p)$ for some $0 < p < 1$.

Exercise 4.* a) Let X be a Poisson random variable with parameter $\lambda > 0$. Compute its characteristic function ϕ_X.

b) Show that for a discrete random variable X with values in $\mathbb{Z}$, the following inversion formula holds:

$$\mathbb{P}(\{X = k\}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-itk}\, \phi_X(t)\, dt, \quad \forall k \in \mathbb{Z}$$

c) Use the above formula to deduce the distribution of the random variable X with values in $\mathbb{Z}$ whose characteristic function is given by

$$\phi_X(t) = \cos(t), \quad t \in \mathbb{R}$$

d) Without solving part c), how could you be sure that ϕ_X is indeed a characteristic function?

Exercise 5. Let $\lambda > 0$ and X be a random variable whose characteristic function ϕ_X is given by $\phi_X(t) = \exp(-\lambda|t|), \quad t \in \mathbb{R}$.

a) What can you deduce on the distribution of X from each of the following facts?

i) ϕ_X is not differentiable in $t = 0$.

ii) $\int_{\mathbb{R}} |\phi_X(t)|\, dt < +\infty$.

b) Use the inversion formula seen in class to compute the distribution of X.

c) Let $Y = \frac{1}{X}$. Using the change of variable formula (not worrying about the fact that X might take the value 0, as this is a negligible event), compute the distribution of Y.

d) Let now $X_1, \ldots, X_n$ be n independent copies of the random variable X. What are the distributions of

$$Z_n = \frac{X_1 + \ldots + X_n}{n} \quad \text{and} \quad W_n = \frac{n}{\frac{1}{X_1} + \ldots + \frac{1}{X_n}} \quad ?$$

e) What oddities do you observe in the results of part d)? (there are at least two)