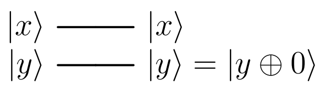
Exercise Set 4: Solution
Quantum Computation

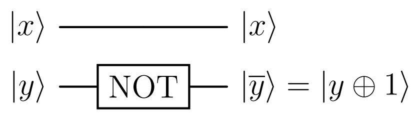
Exercise 1 *Deutsch's algorithm*

(a) The 4 oracle gates U_f are given respectively by:

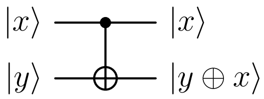
(1) For $f_1(x) = 0$:



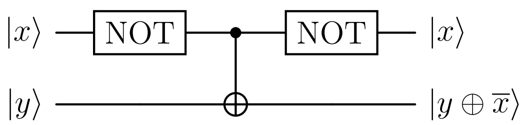
(2) For $f_2(x) = 1$:



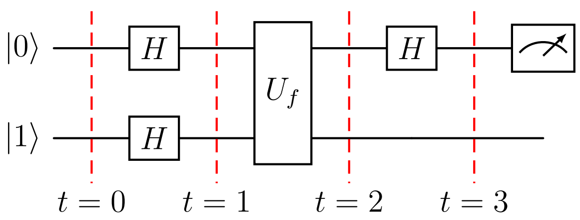
(3) For $f_3(x) = x$:



(4) For $f_4(x) = \overline{x}$:



(b) The Deutsch circuit is the following:



Let us analyze the various states:

- Initially, the state of the 2 qubits is $|\psi_0\rangle = |0\rangle \otimes |1\rangle$.
- After passage through the first Hadamard gates, the state becomes

$$|\psi_1\rangle = H\,|0\rangle \otimes H\,|1\rangle = \frac{1}{2}\left(|00\rangle - |01\rangle + |10\rangle - |11\rangle\right)$$

- After passage through the quantum oracle U_f, the state becomes

$$|\psi_2\rangle = U_f\,|\psi_1\rangle = \frac{1}{2}\left(|0, f(0)\rangle - \left|0, \overline{f(0)}\right\rangle + |1, f(1)\rangle - \left|1, \overline{f(1)}\right\rangle\right)$$

- Then, after passage of the first qubit through the Hadamard gate on the right, the state becomes:

$$\begin{aligned}|\psi_3\rangle = (H \otimes I)\,|\psi_2\rangle &= \frac{1}{2^{3/2}}\Big(|0, f(0)\rangle + |1, f(0)\rangle - \left|0, \overline{f(0)}\right\rangle - \left|1, \overline{f(0)}\right\rangle \\ &\qquad + |0, f(1)\rangle - |1, f(1)\rangle - \left|0, \overline{f(1)}\right\rangle + \left|1, \overline{f(1)}\right\rangle \Big) \\ &= \frac{1}{2^{3/2}}\Big(|0, f(0)\rangle - \left|0, \overline{f(0)}\right\rangle + |0, f(1)\rangle - \left|0, \overline{f(1)}\right\rangle \\ &\qquad + |1, f(0)\rangle - \left|1, \overline{f(0)}\right\rangle - |1, f(1)\rangle + \left|1, \overline{f(1)}\right\rangle \Big)\end{aligned}$$

  after some reordering.
- Let us now analyze the state $|\psi_3\rangle$ in the two cases $f(0) = f(1)$ and $f(0) \neq f(1)$:
  - In the case where $f(0) = f(1) = x$, say, we get:

$$|\psi_3\rangle = \frac{1}{2^{3/2}}\Big(|0, x\rangle - |0, \overline{x}\rangle + |0, x\rangle - |0, \overline{x}\rangle \Big) = \frac{1}{\sqrt{2}}\left(|0, x\rangle - |0, \overline{x}\rangle\right)$$

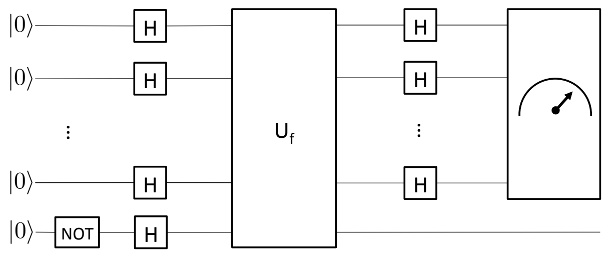
  - In the case where $f(0) = x$ and $f(1) = \overline{x}$, say, we get:

$$|\psi_3\rangle = \frac{1}{2^{3/2}}\Big(|1, x\rangle - |1, \overline{x}\rangle - |1, \overline{x}\rangle + |1, x\rangle \Big) = \frac{1}{\sqrt{2}}\left(|1, x\rangle - |1, \overline{x}\rangle\right)$$

- So finally, measuring the value of the first qubit, we obtain either $|0\rangle$ or $|1\rangle$ (each time with probability 1), which allows us to decide between the two alternatives.

Exercise 2 *Bernstein-Vazirani's algorithm*

(a) We reuse here the same circuit as in the lecture for the Deutsch-Josza algorithm:



The only thing that changes here is the prior information we have on the function f. The output state of the circuit (before the measurement) is given by

$$\begin{aligned}|\psi_4\rangle &= \frac{1}{2^n}\sum_{x\in\{0,1\}^n}\sum_{y\in\{0,1\}^n}(-1)^{f(x)+x\cdot y}\,|y\rangle \otimes \frac{1}{\sqrt{2}}\big(\,|0\rangle - |1\rangle\,\big) \\ &= \sum_{y\in\{0,1\}^n}\left(\frac{1}{2^n}\sum_{x\in\{0,1\}^n}(-1)^{x\cdot(a+y)}\right)|y\rangle \otimes \frac{1}{\sqrt{2}}\big(\,|0\rangle - |1\rangle\,\big)\end{aligned}$$

So after the measurement of the first n qubits, the outcome is state $|y\rangle$ with probability

$$\text{prob}(|y\rangle) = \left| \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot (a+y)} \right|^2$$

which is equal to 1 if $y = a$ and 0 in all the other cases. Therefore the result.

(b) When adding bit b to the picture, we obtain

$$\begin{aligned} \text{prob}(|y\rangle) &= \left| \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{b \oplus x \cdot (a+y)} \right|^2 \\ &= \left| \frac{1}{2^n} (-1)^b \sum_{x \in \{0,1\}^n} (-1)^{x \cdot (a+y)} \right|^2 = \left| \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot (a+y)} \right|^2 \end{aligned}$$

(i) The probabilities remain therefore the same as in the absence of b (which just adds a global phase), so the vector a can be equally determined.

(ii) On the contrary, b remains unknown with this scheme.

Exercise 3 *IBM Q practice: Implementation and tests with the Toffoli gate*

Please refer to the output histograms on Moodle.