Continue on basic defs for tensors.

- As multidimensional arrays of numbers

$$T^{\alpha\beta\gamma} \in \mathbb{R}^{I_1} \times \mathbb{R}^{I_2} \times \mathbb{R}^{I_3}.$$

order $p = 3$

- Mathematical def is as follows: a tensor is a multilinear transformation acting on vectors; Take for example 3 vectors $\underline{x}, \underline{y}, \underline{z}$

$\underline{x} \in \mathbb{R}^{I_1}$, $\underline{y} \in \mathbb{R}^{I_2}$, $z \in \mathbb{R}^{I_3}$:

$$\sum_{\alpha,\beta,\gamma} T^{\alpha\beta\gamma} x^{\alpha} y^{\beta} z^{\gamma} \equiv T(\underline{x}, \underline{y}, \underline{z})$$

$\leftarrow$ scalar $\in \mathbb{R}$.

This expression is linear separately in $\underline{x}$ (d), in $\underline{y}$ (μ), in $\underline{z}$ (e) $\Rightarrow T(d\underline{x}, \mu\underline{y}, e\underline{z})$

$= d\mu e\, T(\underline{x}, \underline{y}, \underline{z})$.

(T is Multilinear transformation).

$$\left[\mathbb{T}(\mathbb{I}, \underline{y}, \underline{z})\right]^{\alpha} = \sum_{\beta,\gamma} T^{\alpha\beta\gamma}\, y^{\beta} z^{\gamma}$$

identity matrix

$\underline{y}$, $\underline{z}$ is a vector $\in \mathbb{R}^{I_1}$.

$\boxed{\mathbb{T}(\mathbb{I}, \underline{y}, \underline{z}) \text{ is bilinear expression}}$

$$\left[\mathbb{T}(\underline{x}, \mathbb{I}, \underline{z})\right]^{\beta} = \sum_{\alpha,\gamma} T^{\alpha\beta\gamma}\, x^{\alpha} z^{\gamma}.$$

a vector $\in \mathbb{R}^{I_2}$

bilinear in $\underline{x}$ & $\underline{z}$.

$$\left[\mathbb{T}(\underline{x}, \mathbb{I}, \mathbb{I})\right]^{\beta\gamma} = \sum_{\alpha} T^{\alpha\beta\gamma}\, x^{\alpha}.$$

linear expression in $\underline{x}$

it is a matrix $\in \mathbb{R}^{I_2} \times \mathbb{R}^{I_3}$.

$$\left[\mathbb{T}(\underbrace{M_1, M_2, M_3}_{\text{3 matrices}})\right]^{\alpha'\beta'\gamma'} = \sum_{\alpha\beta\gamma} T^{\alpha\beta\gamma}\, M_1^{\alpha\alpha'} M_2^{\beta\beta'} M_3^{\gamma\gamma'}$$

obtain trilinear in $\underline{M_1}$, $\underline{M_2}$, $\underline{M_3}$. and is a tensor

Form these multilinear expression to obtain scalars / vectors / matrices / tensors.

- Study of such transformaties is called "Multilinear algebra".

3) <u>Tensor product of vectors.</u>

4) <u>Notion tensor Rank.</u>

↑

important notion in multilinear algebra.

3) Tensor product of vectors.

also called "outer product".

- $\underline{a} \in \mathbb{R}^{I_1}$, $\underline{b} \in \mathbb{R}^{I_2}$ and form the $\underline{a} \otimes \underline{b} \in \mathbb{R}^{I_1} \times \mathbb{R}^{I_2}$ with components $(a \otimes b)^{\alpha\beta} \equiv \underbrace{a^\alpha b^\beta}_{I_1 \times I_2 \text{ matrix.}}$

- $\underline{a} \in \mathbb{R}^{I_1}$, $\underline{b} \in \mathbb{R}^{I_2}$, $\underline{c} \in \mathbb{R}^{I_3}$

  $\underline{a} \otimes \underline{b} \otimes \underline{c} \in \mathbb{R}^{I_1} \times \mathbb{R}^{I_2} \times \mathbb{R}^{I_3}$ with components $(\underline{a} \otimes \underline{b} \otimes \underline{c})^{\alpha\beta\gamma} \equiv \underbrace{a^\alpha b^\beta c^\gamma}_{I_1 \times I_2 \times I_3 \text{ tensor of order 3 (3-mode, 3 way).}}$.

- Generalize to any p.

Remark

$a \otimes b = \underline{a}\, \underline{b}^T$ (column $\times$ line) $\in \mathbb{R}^{I_1} \times \mathbb{R}^{I_2}$. Not to confuse with "inner or scalar product" $a^T \cdot \underline{b} \in \mathbb{R}$.

- Tensor product of matrices:

M with components $M^{\alpha\beta} \in \mathbb{R}^{I_1} \times \mathbb{R}^{I_2}$

N " " $N^{\gamma\delta} \in \mathbb{R}^{I_3} \times \mathbb{R}^{I_4}$

$M \otimes N$ is the tensor with components

$$(M \otimes N)^{\alpha\beta\gamma\delta} = M^{\alpha\beta} N^{\gamma\delta}$$

4-mode tensor in $\mathbb{R}^{I_1} \times \mathbb{R}^{I_2} \times \mathbb{R}^{I_3} \times \mathbb{R}^{I_4}$.

4-dim array of numbers. (hypercube).

#.

Notation: sometimes in reviews or refs (more on the web page)

$\otimes \rightarrow \odot$

preferred for us.
& in math literature.

Later in class other products

Kronecker product
Khatri-Rao product
Hadamard product.

Will be defined as we go on.

4) Tensor Rank.

For two vects $\underline{a}$ & $\underline{b}$ $\rightarrow$ $\underline{a} \otimes \underline{b} = \underbrace{\underline{a}\,\underline{b}^T}_{\text{rank-one matrix}}$.

By analogy we say that

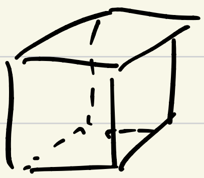
def of Rank one tensors

$\underline{a} \otimes \underline{b} \otimes \underline{c}$ is a rank one tensor / order 3

$\underline{a} \otimes \underline{b} \otimes \underline{c} \otimes \underline{d}$ is a rank one tensor / order 4.

etc.

If I give you an array of numbers $T^{\alpha\beta\gamma}$

$T =$  it is always possible to represent

$$T = \sum_{i} \underline{a}_i \otimes \underline{b}_i \otimes \underline{c}_i$$

$\leftarrow$ decomposition of a tensor.

where $\underline{a}_i, \underline{b}_i, \underline{c}_i$ are vectors, if I allow sufficiently many terms.

Definition of the rank of a tensor.

The rank of a tensor of order p is the minimal (possible) number of terms in decompositions of the form (rank one tensors in the sum).

$$T = \boxed{\sum_{i=1}^{R}} \underline{a}_i^{(1)} \otimes \underline{a}_i^{(2)} \otimes \cdots \otimes \underline{a}_i^{(p)}$$

order p : p - terms in $\otimes$ products.

R : rank if this is the minimal number of terms possible in the sum.

Each term of the sum is a rank-one tensor.

Remarks.

a) For matrices this defn is equivalent to the usual rank defined as the

$$\dim(\text{column spa}) = \dim(\text{row span}).$$

b) For tensors we do not have such a universal equiv definition. Many metrics of rank.

<u>Def introduced before</u>: is called "Tensor Rank"

(Later we will also introduce other metrics of rank.)

c) For tensors it is non-trivial to compute the Rank → will be discussed Next time.

d) For Matrices universal algo to compute the rank is SVD → # of non zero sing values.
but for tensor No such systematic method.

END OF LECT 1.