Tensor Factorisation Methods.

popular algo **Alternating Least Squares Method.**

- Main ideas for today:

Tensor $T^{\alpha\beta\gamma}$ $\rightsquigarrow$ Matrix representation or display of the tensor

highly non unique & we will have to use simultaneously 3 such representations.
$\underline{\underline{3}}$ = order of the tensor.

apply lin alg methods / here apply least square methodology.

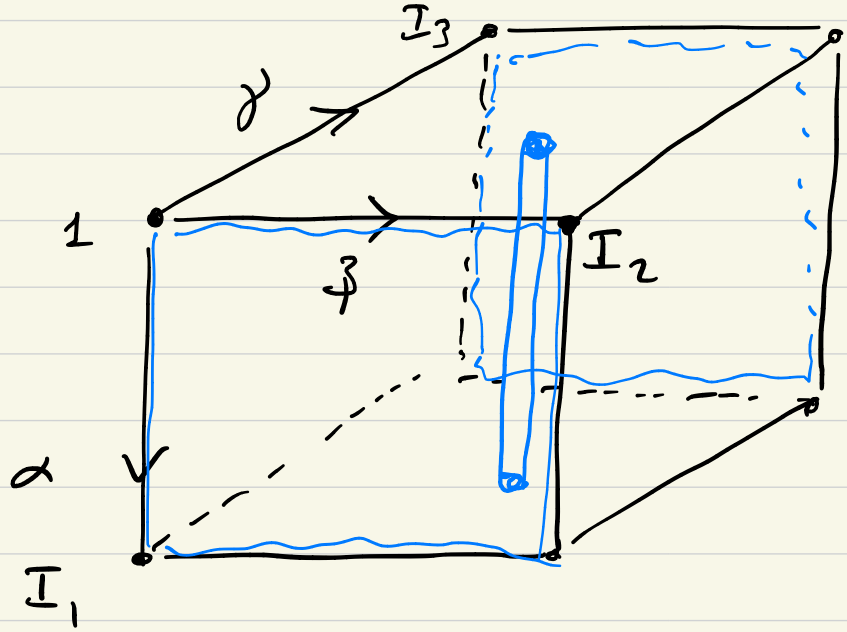
1) Look at Matricisations of a Tensor.

2) Introduce useful products (Kronecker product
Khatri-Rao product
Hadamard product)

3) ALS algorithm.

1) Matricisations of an order 3 tensor.

$$T^{\alpha\beta\gamma}$$

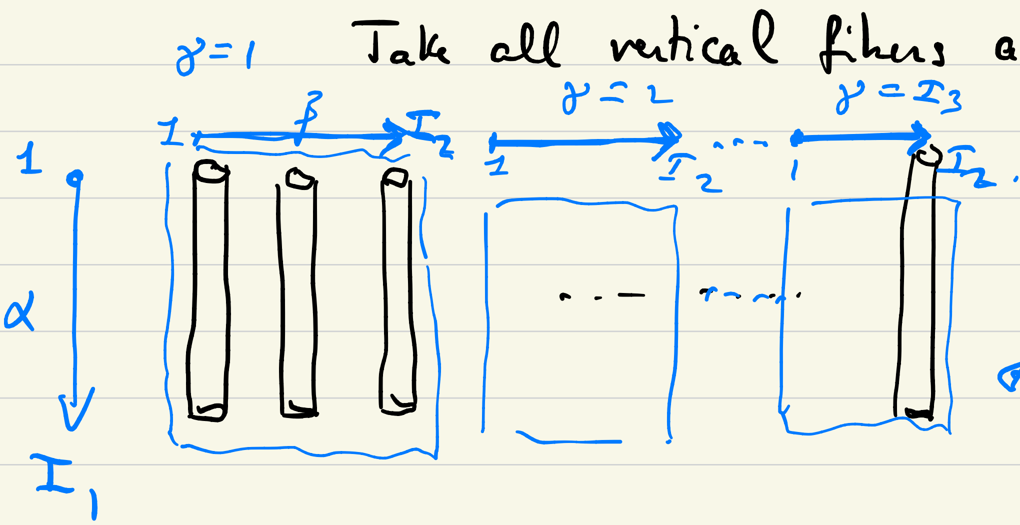


3 Matrix representations of T : $T_{(1)}, T_{(2)}, T_{(3)}$

- first matrix $T_{(1)}$: $\left(T^{\bullet\beta\gamma}\right)_{\alpha=1\ldots I_3}$

for $\beta\gamma$ fixed vector called a "fiber".

Take all vertical fibers and align them

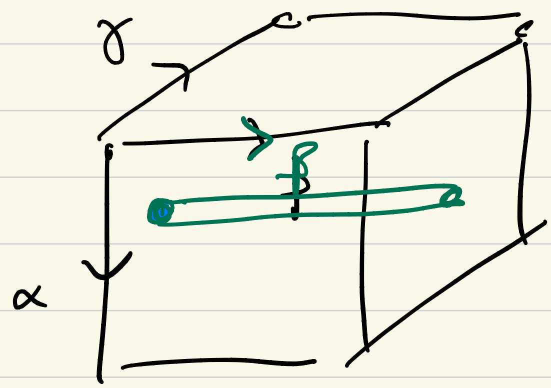


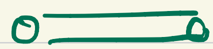
$I_1 \times \underbrace{I_2 I_3}$

$\boxed{\text{Matrix } T_{(1)}}$.

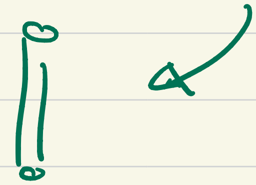
• seconde Matrix $T_{(2)}$: $\left(T^{\alpha \bullet \gamma}\right)_{\beta = 1 \cdots I_2}$

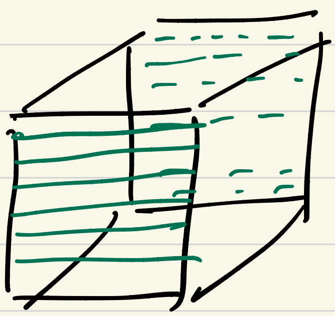
"fiber" or "vector"



horizontal fiber . 

align all horizontal fiber





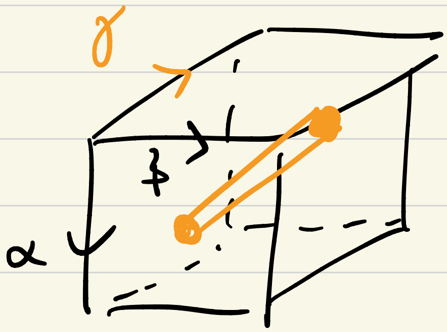
I_2 $\underbrace{}_{I_1}$ ...... $\underbrace{}_{I_1}$ $= T_{(2)} : I_2 \times I_1 I_3$

Matrix

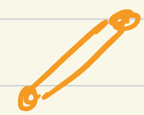
$1 \quad \cdots\cdots \quad I_3$

- Third Matricization $T_{(3)}$ $\quad \left(T^{\alpha\xi\bullet}\right)_{\gamma=1..I_3}$

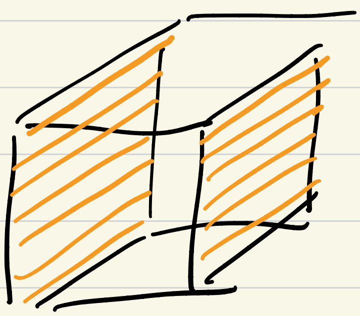
fiber with I_3 components

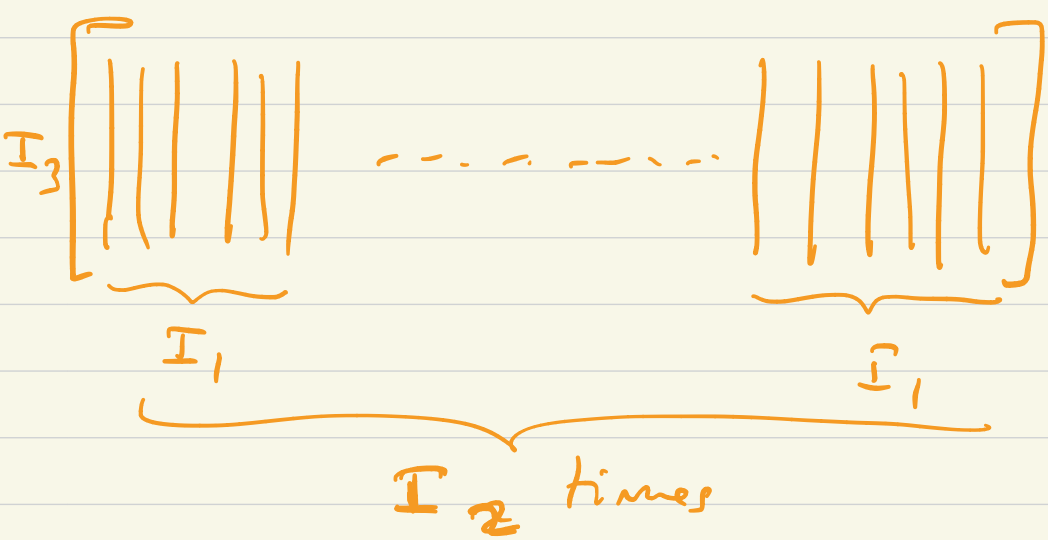


along the depth.



and you align them again





$= T_{(3)}$

Matrix with dimensions

$I_3 \times I_1 I_2$

We will have to understand how to write down Metricization in a more analytic way. $\rightsquigarrow$ Necessary to use some concepts related to tensor product $\otimes$

<u>New products</u>:

a) Kronecker product $\otimes_{Kro}$

b) Khatri-Rao product $\otimes_{Khr}$

c) Hadamard. $*$

(<u>Remark about notation</u>: often $\underset{\text{tensor prod}}{\otimes}$ is denoted $\odot$

and $\otimes_{Kro}$ is denoted by $\otimes$

(this is the case ~~for~~ the review on the web page of class))

a) Kronecker Product : $\underline{c}$ & $\underline{b}$ two column vectors.

dim $\quad I_3 \nearrow \quad \uparrow I_2$

$$\underbrace{\underline{c} \otimes_{kro} \underline{b}}_{\text{Kro pr of two columns}} = \begin{bmatrix} c^1 \\ \vdots \\ c^{I_3} \end{bmatrix} \otimes_{kro} \begin{bmatrix} b^1 \\ \vdots \\ b^{I_2} \end{bmatrix} \equiv \begin{bmatrix} c^1 \underline{b} \\ c^2 \underline{b} \\ \vdots \\ c^{I_3} \underline{b} \end{bmatrix}$$

column vector of dim $I_3 I_2$

(Remember $\underline{c} \otimes \underline{b}$ is a matrix // list of components in $\underline{c} \otimes \underline{b}$ and $\underline{c} \otimes_{kro} \underline{b}$ is the same might say $\underline{c} \otimes_{kro} \underline{b}$ is a "vectorisation" of $\underline{c} \otimes \underline{b}$).

Two properties :

- $(\underline{c} \otimes_{kro} \underline{b})^T = [c^1 \underline{b}^T, \ldots, c^{I_3} \underline{b}^T] \equiv \underbrace{\underline{c}^T \otimes_{kro} \underline{b}^T}_{\text{Kro pr of lines}}$
- $(\underline{e} \otimes_{kro} \underline{d})^T \cdot (\underline{c} \otimes_{kro} \underline{b}) = (\underline{e}^T \cdot \underline{c})(\underline{d}^T \cdot \underline{b})$ ✓

scalar pr $\qquad$ Prove this!

b) Khatri-Rao product

$C = [\underline{c}_1 \; \cdots \; \underline{c}_R]$ $I_3 \times R$ matrix

$B = [\underline{b}_1 \; \cdots \; \underline{b}_R]$ $I_3 \times R$ matrix

$$C \otimes_{khr} B = [\underline{c}_1 \otimes_{kro} \underline{b}_1 ; \underline{c}_2 \otimes_{kro} \underline{b}_2 ; \cdots ; \underline{c}_R \otimes_{kro} \underline{b}_R]$$

$\underline{I_2 I_3 \times R}$ matrix.

c) Hadamard product.

A & B two matrices $(A * B)_{ij} = A_{ij} B_{ij}$.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} * \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11} b_{11} & a_{12} b_{12} \\ a_{21} a_{22} & b_{21} b_{22} \end{pmatrix}.$$

#.

Property (exercise)

$$(E \otimes_{khR} D)^T (C \otimes_{khr} B) = (E^T C) * (D^T B)$$

useful identity to make calculations!

Let us here recall something that was proved in the exercise session:

Lemma: Let C and B have full column rank (all columns of C are lin indep, idem for columns of B).

Then $C \otimes_{Khr} B$ is also full column rank.

i.e $\underline{c}_1 \otimes_{kro} \underline{b}_1 \, ; \, \underline{c}_2 \otimes_{kro} \underline{b}_2 \, ; \, \ldots ; \, \underline{c}_R \otimes_{kro} \underline{b}_R$.

∎

This will be used also for the ALS algorithm.

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Last tool missing is write down some analytic expression for $\boxed{\text{Matricizations}}$ of $\underline{a} \otimes \underline{b} \otimes \underline{c}$ and $\sum_{k=1}^{R} \underline{a}_k \otimes \underline{b}_k \otimes \underline{c}_k$. ?

BREAK