Remarks about HOSVD:

a) Core tensor G is NOT diag (diff than usual matrix SVD)

However $\vec{u}$'s, $\vec{v}$'s, $\vec{w}$'s are $\perp$. (analogous to matrix SVD)

b) Highly NON unique (diff than usual SVD).

c) Discuss low multilinear rank approximation

↑ important in appl for dim red or data compression.

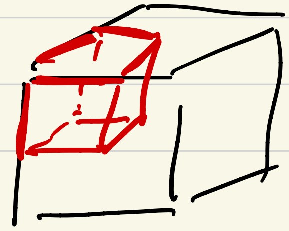
- Recall for matrices Eckart-Young Thm. allows to truncated SVD → Best low rank approx for matrix.

- Here you could hope something similar???

e.g. take truncation of T

$\text{rank}_{\boxplus}(T) = \{R_1, R_2, R_3\}$

$$\tilde{T} = \sum_{p=1}^{K_1} \sum_{q=1}^{K_2} \sum_{r=1}^{K_3} G_{pqr}\, \vec{u}_p \otimes \vec{v}_q \otimes \vec{w}_r .$$



$K_1 < R_1 \,;\; K_2 < R_2 \,;\; K_3 < R_3$

Question is : how do T & $\tilde{T}$ compare???

One can prove that :

$$\|T-\tilde{\tilde{T}}\|_F^2 \leqslant \|T - \tilde{T}_{\{k_1,k_2,k_3\}}\|_F^2 \leqslant C\|T-\tilde{\tilde{T}}\|_F^2$$

where

$$\tilde{\tilde{T}} = \underset{\substack{\operatorname{rank}_{\circledast}(\tilde{\tilde{T}}) \\ =\{k_1,k_2,k_3\}}}{\operatorname{arg\,min}} \|T - \tilde{\tilde{T}}\|_F^2 .$$

- its a pretty good approximation !
- $C = \sqrt{p-1}$

  e.g. $p=3$; $C=\sqrt{2}$.

- The Naive truncation of the core tensor G does Not give the Best Multilinear approx of T.
- However $\tilde{\tilde{T}}$ exist! However dont have systematic algorithms to compute it.

- So all in all the Tucker or HOSVD gives a pretty good approx of T.

4) Last Remark.

Does a truncation of the usual tensor decomp give a good approx of a tensor in any sense?

$$T = \sum_{r=1}^{R} \vec{a}_r \otimes \vec{b}_r \otimes \vec{c}_r \qquad \mathrm{rank}_{\otimes}(T) = R.$$

→ Minimize $\| T - T' \|_F^2$ over tensors T' with $\mathrm{ran}_{\otimes}(T') = K < R$.

→ This problem is not $\boxed{\text{well defined}}$ because the minimum is not attained in space of tensors of $\mathrm{ran}_{\otimes}(T') = K$.

Recall exercise: $W = e_0 \otimes e_1 \otimes e_1 + e_1 \otimes e_0 \otimes e_1 + e_0 \otimes e_0 \otimes e_1$

$e_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\qquad \mathrm{rank}_{\otimes}(W) = 3$.

but $\exists\, D_\epsilon$, $\mathrm{rank}_{\otimes}(D_\epsilon) = 2$ s.t.

$$\lim_{\epsilon \to 0} D_\epsilon = W.$$ jump of rank.

Prove the theorem of Tucker..

· will give us an algorithm for HOSVD.

$T = (T^{\alpha\beta\gamma})$. Consider $\underset{I_1 \times I_2 I_3}{T_{(1)}}$; $T_{(2)}$; $T_{(3)}$

found easily from $T^{\alpha\beta\gamma}$.

Each Matricization has an (matrix) SVD:

$$\bullet \quad T_{(1)} = U^{(1)}_{I_1 \times R_1} \Sigma^{(1)}_{R_1 \times R_1} V^{(1)T}_{R_1 \times I_2 I_3}.$$

because $\mathrm{rank}_{\boxplus}(T) = \{ R_1, R_2, R_3 \}$.

$R_1 = \mathrm{rank}(T_{(1)})$

$$\bullet \quad T_{(2)} = U^{(2)}_{I_2 \times R_2} \Sigma^{(2)}_{R_2 \times R_2} V^{(2)T}_{R_2 \times I_1 I_3}$$

$$\bullet \quad T_{(3)} = U^{(3)}_{I_3 \times R_3} \Sigma^{(3)}_{R_3 \times R_3} V^{(3)T}_{R_3 \times I_1 I_2}$$

• The idea is to consider the following multilinear transform of T:

$$\left[T\left(U^{(1)}, U^{(2)}, U^{(3)}\right) \right]^{pqr} \equiv \sum_{\alpha=1}^{I_1} \sum_{\beta=1}^{I_2} \sum_{\gamma=1}^{I_3} T^{\alpha\beta\gamma} \underbrace{\left(U^{(1)T}\right)_{p\alpha}}_{\equiv} \underbrace{\left(U^{(2)T}\right)_{q\beta}}_{\equiv} \underbrace{\left(U^{(3)T}\right)_{r\gamma}}_{\equiv}$$

$R_1 \times I_1$

$$\equiv G_{pqr} \qquad 1 \leq p \leq R_1, \quad 1 \leq q \leq R_2, \quad 1 \leq r \leq R_3.$$

• What remains to be done is to check that

$$\sum_{p,q,r=1}^{R_1,R_2,R_3} G_{pqr}\, U^{(1)}_{\alpha p}\, U^{(2)}_{\beta q}\, U^{(3)}_{\gamma r} = T^{\alpha\beta\gamma}$$

$U^{(1)}_{\alpha p} \equiv$ I_1-dim vector w.r.t $\alpha = 1 \dots I_1$

claim this sum on the l.h.s is equal to original tensor T.

you recognize here

$$\sum_{p,q,r=1}^{R_1,R_2,R_3} G_{pqr}\, \vec{u}^{(1)}_p \otimes \vec{u}^{(2)}_q \otimes \vec{u}^{(3)}_r = T.$$

the Tucker decomposition HOSVD.

[This Alg is just a linearaly Method.].

Conclusion: summarize Tucker algo for HOSVD.

- input $T^{\alpha\beta\gamma}$ $\qquad \alpha, \beta, \gamma \to \begin{matrix} 1 \cdots I_1 \\ 1 \cdots I_2 \\ 1 \cdots I_3 \end{matrix}$.

- output G_{pqr} $\qquad \left.\begin{matrix} p = 1 \cdots R_1 \\ q = 1 \cdots R_2 \\ r = 1 \cdots R_3 \end{matrix}\right\} \begin{matrix} U = [\vec{u}_1 \cdots \vec{u}_{R_1}] \\ V = [\vec{v}_1 \cdots \vec{v}_{R_2}] \\ W = [\vec{w}_1 \cdots \vec{w}_{R_3}] \end{matrix}$.

orthog arrays of vectors.

1. From $T^{\alpha\beta\gamma}$ look at $T_{(1)}$; $T_{(2)}$; $T_{(3)}$ Matricize one.

2. SVD for $T_{(1)}$; $T_{(2)}$; $T_{(3)}$

R_1 non zero sing values $\qquad$ R_2 Non zero sin values $\qquad$ similar.

$[\vec{u}_1 \cdots \vec{u}_{R_1}] = U$ $\qquad$ $[\vec{v}_1 \cdots \vec{v}_{R_2}] = V$ $\qquad$ $[\vec{w}_1 \cdots \vec{w}_{R_3}] = W$

$T_{(1)} = U \Sigma^{(1)} \tilde{U}^T$ $\qquad$ $T_{(2)} = V \Sigma^{(2)} \tilde{V}^T$

Not used $\qquad$ used $\qquad$ Not used

3. Compute the core tensor $G_{pqr} = \sum_{\alpha\beta\gamma} T^{\alpha\beta\gamma} u_{\alpha p} u_{\beta q} u_{\gamma r}$

4. Finally you have the decomp:

$$T = \sum_{p,q,r} G_{pqr} \, \vec{u}_p \otimes \vec{v}_q \otimes \vec{w}_r .$$