Solutions of the final exam

Please pay attention to the presentation of your answers! **(3 points)**

Exercise 1. Quiz. (21 points)

Answer each yes/no question below (1 pt) and provide a short justification (proof or counter-example) for your answer (2 pts).

a) Let X_1, X_2, X_3 be three random variables, such that for every $1 \leq i \leq 3$, X_i takes values in $\{-1,+1\}$ and $\mathbb{P}(\{X_i = +1\}) = \mathbb{P}(\{X_i = -1\}) = 1/2$. Is it the case that $\sigma(X_1 + X_2, X_1 + X_3) = \sigma(X_1, X_2, X_3)$?

Answer: No, The set $\{X_1 = +1, X_2 = -1, X_3 = -1\}$ does belong to $\sigma(X_1, X_2, X_3)$, but not to $\sigma(X_1 + X_2, X_1 + X_3)$.

b) Let X, Y be two random variables such that $X^2 \geq Y^2$ almost surely, $\mathbb{E}(X) = \mathbb{E}(Y)$ and $\mathbb{E}(X^2) = \mathbb{E}(Y^2)$. Does it necessarily hold that $X = Y$ almost surely ?

Answer: No. Take any non-trivial and square-integrable random variable X and $Y = -X$.

c) Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous cdf and $G : \mathbb{R} \to \mathbb{R}$ be the function defined as

$$G(t) = \begin{cases} 1 - F(1/t) & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

Is it always the case that G is a also a cdf ?

Answer: No. $\lim_{t\to+\infty} G(t) = 1 - F(0)$, and $F(0)$ is not necessarily equal to 0.

d) Let $\phi : \mathbb{R} \to \mathbb{R}$ be the function defined as

$$\phi(t) = \begin{cases} 1 & \text{if } |t| \leq 1 \\ 2 - |t| & \text{if } 1 \leq |t| \leq 2 \\ 0 & \text{if } |t| \geq 2 \end{cases}$$

Is ϕ a characteristic function ?

Hint: Consider the 3×3 matrix $\{A_{ij} = \phi(t_i - t_j)\}_{i,j=1}^3$ with $t_1 = -1$, $t_2 = 0$ and $t_3 = +1$.

Answer: No. $\det(A) = -1 < 0$, so A is not positive semi-definite.

e) Do there exist two non-deterministic and i.i.d. random variables X, Y such that $X + Y$ and $2X$ have the same distribution ?

Answer: Yes. Take two i.i.d. Cauchy random variables (with both infinite variances!).

f) Let $(Y_n,\, n \in \mathbb{N})$ and $(Z_n,\, n \in \mathbb{N})$ be two sequences of random variables. Let us also define, for $n \in \mathbb{N}$:

$$\mathcal{F}_n = \sigma\left(\sum_{j=0}^{k} Y_j\, Z_{k-j},\, 0 \le k \le n\right)$$

Is $(\mathcal{F}_n,\, n \in \mathbb{N})$ a filtration ?

Answer: Yes. Defining $X_n = \sum_{j=0}^{n} Y_j\, Z_{n-j}$, we see that $(\mathcal{F}_n,\, n \in \mathbb{N})$ is the natural filtration of the process X.

g) Let $(X_n,\, n \in \mathbb{N})$ be a sequence of random variables and $(\mathcal{F}_n^X,\, n \in \mathbb{N})$ be its natural filtration. Let also

$$T = \inf\{n \in \mathbb{N} \,:\, X_n > X_{n+1}\}$$

Is T a stopping time with respect to the filtration $(\mathcal{F}_n^X,\, n \in \mathbb{N})$?

Answer: No. The event $\{T = n\}$ depends on X_{n+1} in general and is therefore not $\mathcal{F}_n$-measurable.

Exercise 2. (25 points + BONUS 3 points)

Let $(Z_n,\, n \ge 1)$ be a sequence of i.i.d. $\sim \mathcal{N}(0,1)$ random variables. Let also $a \in \mathbb{R}$ and let $(X_n,\, n \in \mathbb{N})$ be the stochastic process defined recursively as

$$X_0 = 0, \quad X_{n+1} = a\, X_n + Z_{n+1}, \quad n \ge 0$$

Moreover, let $(Y_n,\, n \ge 1)$ be the sequence of random variables defined as

$$Y_n = \sum_{j=0}^{n-1} X_j\, X_{j+1}, \quad n \ge 1$$

a) Compute $\mathbb{E}(Y_n)$ for $n \ge 1$, and when $-1 < a < +1$, compute $\lim_{n\to\infty} \mathbb{E}(Y_n/n)$.

Answer: Let us first observe that $\mathbb{E}(X_n) = 0$; then compute $\mathbb{E}(X_n^2)$ and $\mathbb{E}(X_n X_{n+1})$:

$$\mathbb{E}(X_{n+1}^2) = a^2 \mathbb{E}(X_n^2) + 1, \quad \text{so} \quad \mathbb{E}(X_n) = 1 + a^2 + \ldots + a^{2(n-1)} = \frac{1-a^{2n}}{1-a^2}$$

and

$$\mathbb{E}(X_n X_{n+1}) = a\, \mathbb{E}(X_n^2) = a\, \frac{1-a^{2n}}{1-a^2}$$

Therefore,

$$\mathbb{E}(Y_n) = \frac{a}{1-a^2} \sum_{j=0}^{n-1} (1 - a^{2j}) = \frac{a}{1-a^2} \left(n - \frac{1-a^{2n}}{1-a^2}\right)$$

which in turn implies that

$$\lim_{n\to\infty} \mathbb{E}(Y_n/n) = \frac{a}{1-a^2}$$

From now on, we assume that $\boldsymbol{a = 0}$.

b) Show that

$$\frac{Y_n}{n} \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} 0$$

Answer: Use Chebyshev's inequality:

$$\mathbb{P}\left(\left\{\left|\frac{Y_n}{n}\right| \geq \varepsilon\right\}\right) \leq \frac{\mathbb{E}(Y_n^2)}{n^2\ \varepsilon^2}$$

and compute

$$\mathbb{E}(Y_n^2) = \sum_{j,k=0}^{n-1} \mathbb{E}(X_j X_{j+1} X_k X_{k+1}) = \sum_{j=0}^{n-1} \mathbb{E}(X_j^2)\,\mathbb{E}(X_{j+1}^2) = n-1$$

as only the terms $j = k$ "survive" in the above double sum (by independence of the X_j and the fact that $\mathbb{E}(X_i) = 0$). This leads to the conclusion that

$$\mathbb{P}\left(\left\{\left|\frac{Y_n}{n}\right| \geq \varepsilon\right\}\right) = O\left(\frac{1}{n}\right)$$

i.e. convergence in probability.

c) Show that for $0 \leq s \leq \frac{1}{2}$, it holds that

$$\mathbb{E}(\exp(sY_n)) \leq \frac{1}{\left(\sqrt{1-2s^2}\right)^n}$$

Hint: Condition successively on $\mathcal{F}_j = \sigma(X_1, \ldots, X_j)$, $j = n-1, n-2, \ldots, 1$ and use the inequalities:

$$\mathbb{E}(\exp(szX)) \leq \mathbb{E}(\exp(szX + s^2X^2)) \overset{(*)}{\leq} \frac{1}{\sqrt{1-2s^2}} \exp(s^2z^2)$$

valid for $0 \leq s \leq \frac{1}{2}$, $z \in \mathbb{R}$ and $X \sim \mathcal{N}(0,1)$. Please note that the first inequality is obvious; it is useful in the first step of the computation.

Answer: Conditioning on $\mathcal{F}_{n-1}$, we obtain

$$\mathbb{E}(\exp(sY_n)) = \mathbb{E}(\exp(sY_{n-1})\mathbb{E}(\exp(sX_{n-1}X_n)|\mathcal{F}_{n-1})) = \mathbb{E}(\exp(sY_{n-1})\varphi(X_{n-1}))$$

where $\varphi(z) = \mathbb{E}(\exp(szX_n)) \leq \frac{\exp(s^2z^2)}{\sqrt{1-2s^2}}$ by the hint. The next step gives

$$\begin{aligned}
&\mathbb{E}(\exp(sY_n)) \leq \frac{1}{\sqrt{1-2s^2}}\,\mathbb{E}(\exp(sY_{n-1} + s^2X_{n-1}^2)) \\
&= \frac{1}{\sqrt{1-2s^2}}\,\mathbb{E}(\exp(sY_{n-2})\,\mathbb{E}(\exp(sX_{n-2}X_{n-1} + s^2X_{n-1}^2)|\mathcal{F}_{n-2})) \\
&= \frac{1}{\sqrt{1-2s^2}}\,\mathbb{E}(\exp(sY_{n-2})\psi(X_{n-2}))
\end{aligned}$$

where $\psi(z) = \mathbb{E}(\exp(szX_{n-1} + s^2X_{n-1}^2)) \leq \frac{\exp(s^2z^2)}{\sqrt{1-2s^2}}$, again by the hint. Proceeding recursively, we reach the desired conclusion (remember that $x_0 = 0$).

BONUS: Prove the last inequality $(*)$.

Answer:

$$\mathbb{E}(\exp(szX + s^2X^2)) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \, \exp\left(-\frac{x^2}{2} + szx + s^2x^2\right) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \, \exp\left(-\frac{1}{2}(1-2s^2)x^2 + szx\right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \, \exp\left(-\frac{1}{2}\left(\sqrt{1-2s^2}\, x + \frac{sz}{\sqrt{1-2s^2}}\right)^2 + \frac{1}{2}\frac{s^2z^2}{1-2s^2}\right) = \frac{1}{\sqrt{1-2s^2}} \exp\left(\frac{1}{2}\frac{s^2z^2}{1-2s^2}\right)$$

$$\leq \frac{\exp(s^2z^2)}{\sqrt{1-2s^2}} \quad \text{for } s \leq 1/2$$

d) Deduce from there that for every $t > 0$, there exists a constant $C > 0$ (possibly depending on t) such that for every $n \geq 1$,

$$\mathbb{P}(\{Y_n \geq nt\}) \leq \exp(-Cn)$$

Hint: In order to simplify computations, you may use the inequality $-\log(1-x) \leq 2x$, valid for $0 \leq x \leq \frac{1}{2}$.

Answer: As mentioned during the exam, the above hint should be ignored. By Chebyshev's inequality and c), we have for $0 \leq s \leq 1/2$:

$$\mathbb{P}(\{Y_n \geq nt\}) \leq \frac{\mathbb{E}(\exp(sY_n))}{\exp(snt)} \leq \frac{\exp(-snt)}{\left(\sqrt{1-2s^2}\right)^n} = \exp(-n(st - \tfrac{1}{2}\log(1-2s^2))$$

Various arguments here allow to conclude: notably, one can observe graphically that for every $t > 0$, there exists $0 < s \leq 1/2$ depending on t such that

$$C = st - \tfrac{1}{2}\log(1-2s^2) > 0$$

e) Is the process $(Y_n,\, n \geq 1)$ a martingale with respect to $(\mathcal{F}_n,\, n \geq 1)$? Justify.

Answer: Yes. Indeed, for each n, Y_n is integrable because the Z_j are Gaussian random variables, Y_n is $\mathcal{F}_n$-measurable, and

$$\mathbb{E}(Y_{n+1}|\mathcal{F}_n) = Y_n + X_n\, \mathbb{E}(X_{n+1}) = Y_n + 0 = Y_n$$

Exercise 3. (25 points)

Let $0 < \sigma < 1$ and $(Z_n,\, n \in \mathbb{N})$ be a collection of i.i.d. and zero-mean random variables such that $\mathrm{Var}(Z_1) = \sigma^2$ and $|Z_n(\omega)| \leq 1$ for all $n \in \mathbb{N}$ and $\omega \in \Omega$. Let also X, Y be the two stochastic processes defined recursively as

$$\begin{cases} X_0 = 1, \quad X_{n+1} = X_n\,(1 + Z_{n+1}), \quad n \geq 0 \\ Y_0 = 1, \quad Y_{n+1} = Y_n\,(1 - Z_{n+1}), \quad n \geq 0 \end{cases}$$

a) Compute recursively $\mathrm{Cov}(X_n, Y_n)$ for $n \in \mathbb{N}$. Is this covariance increasing or decreasing with n ?

Answer: $\mathrm{Cov}(X_n, Y_n) = \mathbb{E}(X_n Y_n) - \mathbb{E}(X_n)\,\mathbb{E}(Y_n)$, $\mathbb{E}(X_n) = \mathbb{E}(Y_n) = 1$ and

$$\mathbb{E}(X_{n+1}Y_{n+1}) = \mathbb{E}(X_n Y_n)\,\mathbb{E}(1 - Z_{n+1}^2) = \mathbb{E}(X_n Y_n)\,(1-\sigma^2)$$

so $\mathbb{E}(X_n Y_n) = (1-\sigma^2)^n$ and $\mathrm{Cov}(X_n, Y_n) = (1-\sigma^2)^n - 1$, decreasing in n.

Now, let $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_n = \sigma(Z_1, \dots, Z_n)$ for $n \geq 1$, and

$$M_n = X_n + Y_n, \quad n \geq 0$$

b) Does it hold that $(M_n,\, n \in \mathbb{N})$ is a Markov process with respect to the filtration $(\mathcal{F}_n,\, n \in \mathbb{N})$, i.e. that

$$\mathbb{E}(g(M_{n+1})\,|\,\mathcal{F}_n) = \mathbb{E}(g(M_{n+1})\,|\,M_n)$$

for every $n \geq 0$ and $g \in C_b(\mathbb{R})$? Justify.

Answer: No. $M_{n+1} = X_{n+1} + Y_{n+1} = (X_n + Y_n) + (X_n - Y_n)\,Z_{n+1} = M_n + (X_n - Y_n)\,Z_{n+1}$. The random variable M_{n+1} can therefore not be written as a simple function of M_n and Z_{n+1}; the process M is not a Markov process.

c) Show that the process $(M_n,\, n \in \mathbb{N})$ is a martingale with respect to the filtration $(\mathcal{F}_n,\, n \in \mathbb{N})$.

Answer: By induction, M_n is integrable and $\mathcal{F}_n$-measurable. Also

$$\mathbb{E}(M_{n+1}|\mathcal{F}_n) = X_n\,\mathbb{E}(1 + Z_{n+1}) + Y_n\,\mathbb{E}(1 - Z_{n+1}) = X_n + Y_n = M_n$$

d) Compute the process $(A_n,\, n \in \mathbb{N})$ defined recursively as

$$A_0 = 0, \quad A_{n+1} = A_n + \mathbb{E}(M_{n+1}^2\,|\,\mathcal{F}_n) - M_n^2, \quad n \geq 0$$

Answer:

$$A_{n+1} - A_n = M_n^2 + 0 + (X_n - Y_n)^2\,\mathbb{E}(Z_{n+1}^2) - M_n^2 = \sigma^2\,(X_n - Y_n)^2$$

so $A_n = \sigma^2 \sum_{j=0}^{n-1} (X_j - Y_j)^2$.

e) What do you know about the process $(M_n^2 - A_n,\, n \in \mathbb{N})$?

Answer: It is a martingale.

f) Does there exist a random variable M_∞ such that $M_n \underset{n\to\infty}{\to} M_\infty$ almost surely ? Justify.

Answer: Yes, as M is a non-negative martingale (second version of the martingale convergence theorem).

g) Does there exist a random variable M_∞ such that $M_n \underset{n\to\infty}{\overset{L^2}{\to}} M_\infty$? Justify.

Answer: No, as $\mathrm{Var}(M_n) = \mathrm{Var}(X_n + Y_n) = \mathrm{Var}(X_n) + \mathrm{Var}(Y_n) + 2\mathrm{Cov}(X_n, Y_n)$ and by induction, we have

$$\mathrm{Var}(X_n) = \mathrm{Var}(Y_n) = (1+\sigma^2)^n$$

and $\text{Cov}(X_n, Y_n) \to -1$ and $n \to \infty$ by part a). So $\text{Var}(M_n)$ goes to $+\infty$; L^2-convergence cannot happen.

Exercise 4. (16 points)

Let $M = (M_n,\, n \in \mathbb{N})$ be a stochastic process defined recursively as follows:

$$M_0 = x < 0, \quad M_{n+1} = \begin{cases} \dfrac{3M_n + 1}{2} & \text{with probability } \frac{1}{2} \\[2ex] \dfrac{M_n}{2} & \text{with probability } \frac{1}{2} \end{cases}$$

a) Show that the process $(M_n,\, n \in \mathbb{N})$ is submartingale.

Answer: For each $n \in \mathbb{N}$, M_n is integrable, as it takes only a finite number fo values, and

$$\mathbb{E}(M_{n+1}|\mathcal{F}_n) = \frac{1}{2}\,\frac{3M_n + 1}{2} + \frac{1}{2}\,\frac{M_n}{2} = M_n + \frac{1}{4} \geq M_n$$

(where $(\mathcal{F}_n,\, n \in \mathbb{N})$ is the natural filtration of M).

Let us now consider the stopping time $T = \inf\{n \geq 1 \,:\, M_n \geq 0\}$, as well as the stopped submartingale $N = M^T$ defined as

$$N_n = M_n^T = M_{T \wedge n} = M_{\min(T,n)} \quad \text{for } n \in \mathbb{N}$$

b) Explain why there exists a random variable N_∞ such that $N_n \underset{n\to\infty}{\to} N_\infty$ almost surely.

Answer: N is a submartingale, which by definition is bounded above by the value $1/2$, so

$$\sup_{n \in \mathbb{N}} \mathbb{E}(N_n^+) \leq \frac{1}{2} < +\infty$$

and by the second version of the martingale convergence theorem (to be more precise: its generalization to submartingales), N_∞ exists.

c) Does it hold that $\mathbb{E}(N_\infty|\mathcal{F}_n) = N_n$ for every $n \in \mathbb{N}$? Justify.

Answer: No. As N is a (strict) submartingale, it will hit 0 with probability 1, so $N_\infty \geq 0$, but $N_0 = x < 0$ by assumption, which makes the above equality impossible, at least for $n = 0$.

d) To what interval in $\mathbb{R}$ does the random variable N_∞ belong ?

Remark: Of course, $\mathbb{R}$ itself is a valid answer to the previous question, but we are actually asking here for the interval of minimal size to which N_∞ is guaranteed to belong.

Answer: From the definition of T and the observation that from a negative value, N_n cannot move above $+1/2$ (excluded), we obtain that $N_\infty \in [0, 1/2[$.