MCAA lecture 4

Let $X = (X_n, n \geq 0)$ be a Markov chain with state space S and transition matrix P.

Theorem (without proof)

Assume X is irreducible.

Then X is positive-recurrent iff it admits a stationary distribution π; in this case, π is unique.

Ergodic theorem

Assume X is ergodic (i.e. irred., aper. & positive-recurrent)

Then X admits a unique limiting & stationary distribution π.

Tools for the proof:

1. Total variation distance between two distributions

Def: Let μ & ν be two distributions on the same state space S (i.e. $0 \leq \mu_i, \nu_i \leq 1$, $\sum_{i\in S} \mu_i = \sum_{i \in S} \nu_i = 1$)

The total variation distance between μ & ν is:

$$\|\mu - \nu\|_{TV} = \sup_{A \subset S} |\mu(A) - \nu(A)| \qquad \begin{cases} \mu(A) = \sum_{i\in A} \mu_i \\ \nu(A) = \sum_{i \in A} \nu_i \end{cases}$$

Properties:

- $0 \leq \|\mu - \nu\|_{TV} \leq 1$

0 ↑ $\mu = \nu$

1 ↑ μ & ν have disjoint support $\left(\exists A \subset S \text{ s.t. } \mu(A) = 1 \text{ \& } \nu(A) = 0 \right)$

- triangle inequality: $\|\mu - \pi\|_{TV} \leq \|\mu - \nu\|_{TV} + \|\nu - \pi\|_{TV}$
- $\|\mu - \nu\|_{TV} = \frac{1}{2} \sum_{i \in S} |\mu_i - \nu_i|$

2. Coupling between two distributions

Def: Let μ, ν be two distributions on S. A coupling between μ & ν is a pair of random variables (X, Y) with a joint distribution on $S \times S$ such that
$\mathbb{P}(X = i) = \mu_i \quad \forall i \in S$ & $\mathbb{P}(Y = j) = \nu_j \quad \forall j \in S$

Rmk: For a given pair (μ, ν), there are multiple couplings!

Example:

$S = \{0, 1\}$ $\quad \mu_0 = \mu_1 = \frac{1}{2}$ $\quad \nu_0 = \nu_1 = \frac{1}{2}$

a) Choose X, Y independently with $\mathbb{P}(X=i, Y=j) = \frac{1}{4}$ $\forall i, j \in S$

b) Choose X, Y such that $\mathbb{P}(X=Y=0) = \mathbb{P}(X=Y=1) = \frac{1}{2}$

In this case, $X = Y$.

c) Choose X, Y such that $\mathbb{P}(X=0, Y=1) = \mathbb{P}(X=1, Y=0) = \frac{1}{2}$

Proposition

For <u>every</u> coupling (X, Y) of μ & ν, we have:

$$\|\mu - \nu\|_{TV} \leq \mathbb{P}(X \neq Y)$$

Proof

Let A be any subset of S:

$$\mu(A) = \mathbb{P}(X \in A) = \mathbb{P}(X \in A, Y \in A) + \mathbb{P}(X \in A, Y \in A^c)$$

$$\nu(A) = \mathbb{P}(Y \in A) = \mathbb{P}(X \in A, Y \in A) + \mathbb{P}(X \in A^c, Y \in A)$$

$$\mu(A) - \nu(A) = \mathbb{P}(X \in A, Y \in A^c) - \mathbb{P}(X \in A^c, Y \in A)$$

$$\leq \mathbb{P}(X \in A, Y \in A^c) \leq \mathbb{P}(X \neq Y)$$

$$\nu(A) - \mu(A) = \mathbb{P}(X \in A^c, Y \in A) - \mathbb{P}(X \in A, Y \in A^c)$$

$$\leq \mathbb{P}(X \in A^c, Y \in A) \leq \mathbb{P}(X \neq Y)$$

So $\forall A \subset S$, $|\mu(A) - \nu(A)| \leq \mathbb{P}(X \neq Y)$

So $\|\mu - \nu\|_{TV} = \sup_{A \subset S} |\mu(A) - \nu(A)| \leq \mathbb{P}(X \neq Y)$ #

Coupling of Markov chains

Let X, Y be two Markov chains defined on the same state space S and with the same transition matrix P, but with initial distributions $\pi^{(0)} = \mu$ and $\pi^{(0)} = \nu$, respectively.
The distributions of these two chains are given by:

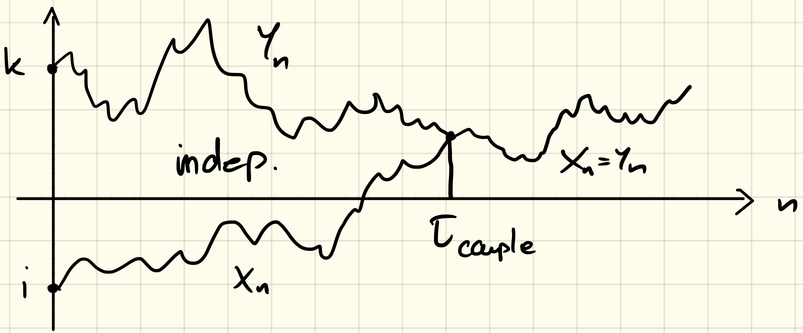
$$\mathbb{P}(X_n = i) = (\mu \cdot P^n)_i \qquad \mathbb{P}(Y_n = i) = (\nu \cdot P^n)_i \qquad i \in S$$

Coupling of X and Y (one possibility):

Let $(Z_n = (X_n, Y_n), n \geq 0)$ be the process on the state space $S \times S$ defined as:

- $\mathbb{P}(Z_0 = (i, k)) = \mu_i \cdot \nu_k \quad i, k \in S$

- X, Y evolve independently (according to P) as long as $X_n \neq Y_n$ ("statistical coupling")
- As soon as $X_n = Y_n$, then the two processes coalesce i.e. $X_m = Y_m$ $\forall m \geq n$, and they evolve together according to P. ("grand coupling")



<u>Def</u>: $\tau_{couple} = \inf \{ n \geq 1 : X_n = Y_n \}$ coupling time

Lemma: $$\| \underbrace{\mu P^n}_{\text{dist. of } X_n} - \underbrace{\nu P^n}_{\text{dist of } Y_n} \|_{TV} \leq \mathbb{P}(\tau_{\text{couple}} > n)$$

Proof:
- $\mu P^n, \nu P^n$ are both distributions on S
- $Z_n = (X_n, Y_n)$ is a coupling of these two distributions
- by the proposition above:

$$\| \mu P^n - \nu P^n \|_{TV} \leq \mathbb{P}(X_n \neq Y_n) = \mathbb{P}(\tau_{\text{couple}} > n)$$

because of our choice of coupling!

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Proof of the ergodic theorem

<u>Reminder</u>: to be proven : $\overset{\text{stat. dist.}}{\overset{\shortparallel}{\pi}}$ is a <u>limiting</u> distribution, i.e.

$$\forall \pi^{(0)}, \quad \lim_{n\to\infty} \pi_i^{(n)} = \pi_i \quad \forall i \in S$$

Actually, we will prove the slightly stronger statement: (when $|S| = +\infty$)

$$\forall \pi^{(0)}, \quad \lim_{n\to\infty} \underbrace{\| \pi^{(n)} - \pi \|_{TV}}_{= \frac{1}{2} \sum_{i\in S} |\pi_i^{(n)} - \pi_i|} = 0$$

Let X be the chain with trans. matrix P & init. dist. $\pi^{(0)}$

Y be the chain " " & init. dist. π

Then $\pi_i^{(n)} = \mathbb{P}(X_n = i) = \left(\pi^{(0)} \cdot P^n\right)_i$

$$\mathbb{P}(Y_n = i) = \left(\pi \cdot P^n\right)_i = \pi_i$$

So by the lemma:

$$\|\pi^{(n)} - \pi\|_{TV} = \|\left(\pi^{(0)} P^n\right) - \left(\pi P^n\right)\|_{TV}$$

$$\leq \mathbb{P}(X_n \neq Y_n) = \mathbb{P}(\tau_{\text{couple}} > n) \xrightarrow[n\to\infty]{?} 0$$

to be proven!

This is equivalent to proving that $\mathbb{P}(\tau_{\text{couple}} < +\infty) = 1$:

Indeed: $\mathbb{P}(\tau_{\text{couple}} < +\infty) = 1 - \mathbb{P}(\tau_{\text{couple}} = +\infty)$

$$= 1 - \mathbb{P}\Big(\bigcap_{n \geq 1} \{\tau_{\text{couple}} > n\}\Big) = 1 - \lim_{n\to\infty} \mathbb{P}(\tau_{\text{couple}} > n)$$

So $\boxed{\mathbb{P}(\tau_{\text{couple}} < +\infty) = 1}$ iff $\lim_{n\to\infty} \mathbb{P}(\tau_{\text{couple}} > n) = 0$ #

Let $(Z_n = (X_n, Y_n),\ n \geq 0)$ be the coupled chain <u>before coalescence</u> (only statistical coupling).

<u>Step 1</u>: <u>Z is positive-recurrent</u>

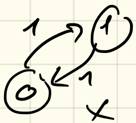
- Z is a Markov chain on the state space $S \times S$ with transition probabilities:

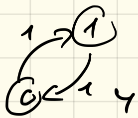
$$\mathbb{P}\left(Z_{n+1} = (j, \ell) \mid Z_n = (i, k) \right) = p_{ij} \cdot p_{k\ell} \qquad (\text{indep.})$$

- Z is <u>irreducible</u>:

⚠ it is not only because X, Y are irreducible !

<u>Ctr-ex</u>:





$Z : (0,0) \to (1,1) \to (0,0) \to \ldots$

Fact: $\begin{cases} \text{If a chain } X \text{ is irreducible and aperiodic, then} \\ \forall i, j \in S, \ \exists N(i,j) \geq 1 \text{ s.t. } p_{ij}(n) > 0 \quad \forall n \geq N(i,j) \end{cases}$

$\Big($ Comes from: if $\gcd(a,b) = 1$, then $\{na + mb;\ n, m \geq 1\}$ contains $\{N, N+1, N+2, \dots\}$ for some $N \geq 1$ $\Big)$

Thus, for Z, we have:

$\forall (i,k), (j,\ell) \in S \times S$, $\exists N(ik, j\ell) = \max\big(N(i,j), N(k,\ell)\big)$

such that $\mathbb{P}(Z_n = (j\ell) \mid Z_0 = (ik)) = \underbrace{p_{ij}(n)}_{>0} \cdot \underbrace{p_{k\ell}(n)}_{>0} \quad \forall n \geq N(ik, j\ell)$

So Z is irreducible and aperiodic.

- Z admits a <u>stationary distribution</u>: $\Pi_{ik} = \pi_i \cdot \pi_k \quad \forall i,k \in S$

$$\sum_{i,k \in S} \Pi_{ik} (P_{ij} \cdot P_{k\ell}) = \sum_{i \in S} \pi_i \, P_{ij} \cdot \sum_{k} \pi_k \, P_{k\ell} = \pi_j \cdot \pi_\ell$$
$$= \Pi_{j\ell}$$

✓

- By the first theorem, Z is <u>positive-recurrent</u>.

<u>Step 2</u>: $\mathbb{P}\left(\tau_{\text{couple}} < +\infty \mid Z_0 = (j\ell) \right) = 1 \quad \forall j, \ell \in S$

Z is positive-recurrent means: $\mathbb{P}\left(T_{(ik)} < +\infty \mid Z_0 = (ik) \right) = 1 \quad \forall i,k \in S$

$$T_{(ik)} = \inf \{ n \geq 1 : Z_n = (ik) \}$$

This implies that $\mathbb{P}\left(T_{(ik)} < +\infty \mid Z_0 = (j\ell) \right) = 1 \quad \forall i, k, j, \ell \in S$

Indeed: $\underbrace{\mathbb{P}\big(T_{(ik)}=+\infty \mid Z_0=(ik)\big)}=0$ $\quad (\Longleftrightarrow \text{pos-rec.})$

$$\geq \mathbb{P}\big(T_{(ik)}=+\infty,\ Z_n=(j\ell) \mid Z_0=(ik)\big)$$

consider n s.t. $p_{ik,j\ell}(n)>0$

$$= \underbrace{\mathbb{P}\big(T_{(ik)}=+\infty \mid Z_n=(j\ell), \cancel{Z_0=(ik)}\big)} \cdot \underbrace{p_{ik,j\ell}(n)}_{>0}$$

$$= \mathbb{P}\big(T_{(ik)}=+\infty \mid Z_0=(j\ell)\big)$$ time-homogeneity

So $\mathbb{P}\big(T_{(ik)}=+\infty \mid Z_0=(j\ell)\big)=0 \quad \forall i,j,k,\ell\in S$.

Now, consider $i=k$: $\mathbb{P}\big(\underline{\underline{T_{(ii)}}}<+\infty \mid Z_0=(j\ell)\big)=1 \quad \forall i,j,\ell\in S$

As $T_{\text{couple}} \leq T_{(ii)}$ $\forall i\in S$, this implies that

$$\mathbb{P}\big(T_{\text{couple}}<+\infty \mid Z_0=(j,\ell)\big)=1 \quad \forall j,\ell\in S$$

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