For example, if a top. mnfd M can be covered by a single chart, then the smooth compatibility condition is trivially satisfied, so any such chart determines automatically a smooth structure on M (see, e.g., Examples 1.3(1) and 1.8(1))

· <u>DEF. 1.7</u>: Let M be a smooth mnfd. Any chart (U,φ) contained in the maximal smooth atlas is called a <u>smooth chart</u>. The corresponding coordinate map φ is called a <u>smooth coordinate map</u>, and its domain U is called a <u>smooth coordinate domain</u>, or <u>smooth coordinate neighborhood</u> of each of its pts.

⇝ smooth coordinate ball

⇝ smooth coordinate cube

· <u>EXAMPLE 1.8</u>:

0) For each $n \in \mathbb{N}$ the Euclidean space $\mathbb{R}^n$ is a smooth n-mnfd with the smooth structure determined by the atlas $\{(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n})\}$. We call this the <u>standard smooth structure on $\mathbb{R}^n$</u> and the resulting coordinate map <u>standard coordinates</u>. W.r.t. this smooth structure, the smooth coordinate charts for $\mathbb{R}^n$ are exactly those charts (U,φ) s.t. φ is a diffeo. (in the usual sense) from $U \subseteq \mathbb{R}^n$ to another open subset $\hat{U} \subseteq \mathbb{R}^n$.

1) <u>Graphs of smooth fncts</u>: If $U \subseteq \mathbb{R}^n$ is an open subset and if $f: U \to \mathbb{R}^k$ is a smooth fnct, then by EX. 1.3(1) the graph

$\Gamma(f)$ of f is a top. n-mnfd in the subspace top. Since $\Gamma(f)$ is covered by the single graph coordinate chart $\varphi : \Gamma(f) \to U$, we can put a canonical smooth structure on $\Gamma(f)$ by declaring $(\Gamma(f), \varphi)$ to be a smooth chart.

2) Spheres : The n-sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ is a top. n-mnfd by EX. 1.3(2). We put a smooth structure on $\mathbb{S}^n$ as follows. For each $i \in \{1, \dots, n+1\}$ we consider the graph coordinate charts $(U_i^{\pm} \cap \mathbb{S}^n, \varphi_i^{\pm})$. For any $i \neq j$ and any choice of $\pm$ signs, the transition map $\varphi_i^{\pm} \circ (\varphi_j^{\pm})^{-1}$ and $\varphi_i^{\pm} \circ (\varphi_j^{\mp})^{-1}$ are easily computed. For example, when $i<j$, we get :

$$\varphi_i^{+} \circ (\varphi_j^{+})^{-1}(u^1, \dots, u^n) = \varphi_i^{+}\left(u^1, \dots, \sqrt{1-|u|^2}, \dots, u^n\right)$$
$$= \Big(u^1, \dots, \underset{\substack{\downarrow \\ i\text{-th}}}{\widehat{u}^i}, \dots, \underset{\substack{\downarrow \\ j\text{-th}}}{\sqrt{1-|u|^2}}, \dots, u^n\Big),$$

and similar formular hold in the other cases. When $i=j$, the domains of φ_i^{+} and φ_i^{-} are disjoint, so there is nothing to check. Thus, the collection of charts $\{(U_i^{\pm} \cap \mathbb{S}^n, \varphi_i^{\pm})\}_{i=1}^{n+1}$ is a smooth atlas, so it defines a smooth structure on $\mathbb{S}^n$, which we call its standard smooth structure.

3) Open submanifolds : Let U be any open subset of $\mathbb{R}^n$. Then U is a top. n-mnfd, and the single chart (U, Id_U) determines a smooth structure on U.

More generally, let M be a smooth n-mnfd and let $U \subseteq M$ be an open subset. Define an atlas on U by

$\mathcal{A}_U = \{$smooth charts (V, φ) for M s.t. $V \subseteq U\}$.

Every pt $p \in U$ is contained in the domain of some chart (W, φ) for M; if we set $V = W \cap U$, then $(V, \varphi|_V)$ is a chart in $\mathcal{A}_U$ whose domain contains p. Therefore, U is covered by the domains of the charts in $\mathcal{A}_U$, and it is easy to verify that this is a smooth atlas for U.

Thus, any open subset of M is itself a smooth n-mnfd in a natural way. Endowed with this smooth structure, we call any open subset an <u>open submanifold of M</u>.

In the examples we have seen so far, we constructed a smooth manifold structure in two stages: we started with a top. sp. and checked that it was a top. mnfd, and then we specified a smooth structure. The following lemma shows how, given a set with suitable "charts" that overlap smoothly, we can use the charts to define both a topology and a smooth structure on the set.

· <u>LEM 1.9</u> (Smooth mnfd chart lemma): Let M be a set. Suppose we are given a collection $\{U_\alpha\}$ of subsets of M together with maps $\varphi_\alpha : U_\alpha \longrightarrow \mathbb{R}^n$ s.t. the following prop. are satisfied:

(i) For each α, φ_α is a bijection between U_α and an open subset $\varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$.

(ii) For each α and β, the sets $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$

are open in $\mathbb{R}^n$.

(iii) Whenever $U_\alpha \cap U_\beta \neq \emptyset$, the map

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$$

is smooth.

(iv) Countably many of the sets U_α cover M.

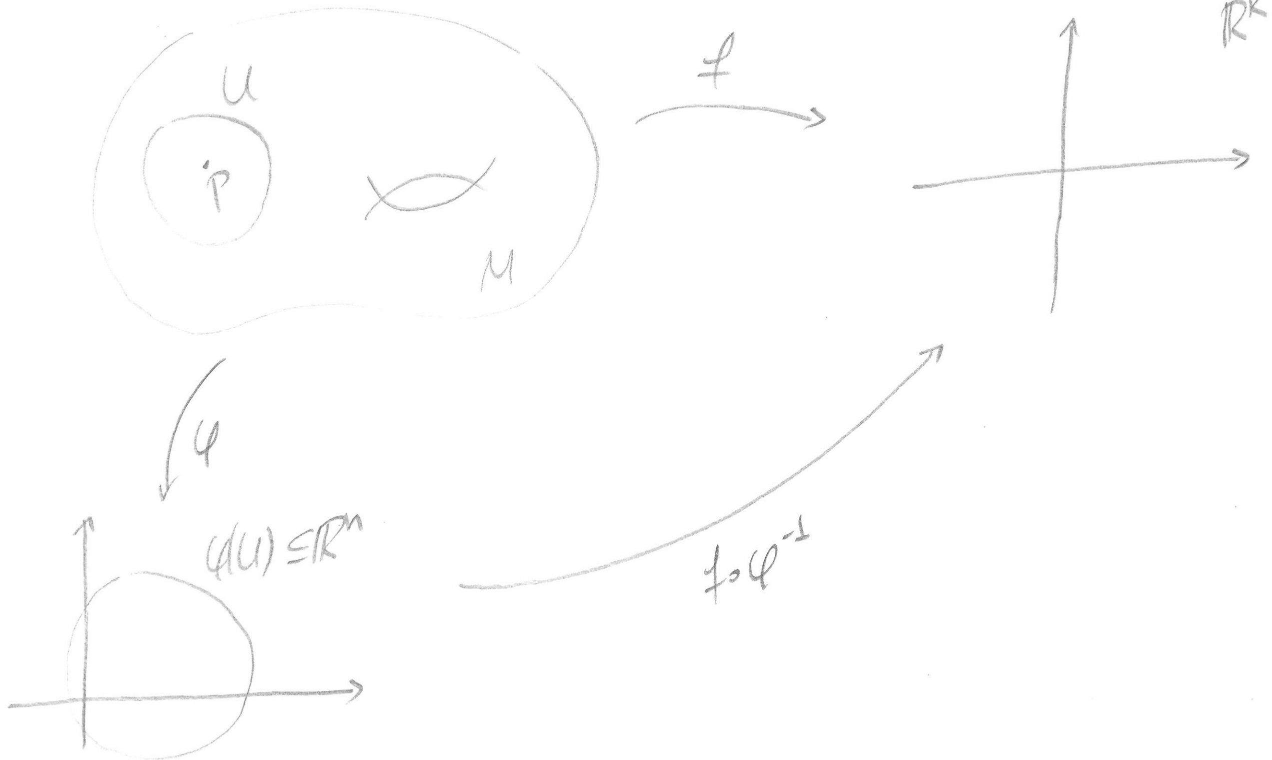
(v) Whenever $p, q \in M$ with $p \neq q$, either there exists some U_α containing both p and q or there exist disjoint sets U_α and U_β with $p \in U_\alpha$ and $q \in U_\beta$.

Then M has a unique manifold structure s.t. each $(U_\alpha, \varphi_\alpha)$ is a smooth chart.

- details of the proof : [Lee, Lemma 1.35]
- key idea of the proof : define the topology on M by taking all sets of the form $\varphi_\alpha^{-1}(V)$, $V \subseteq \mathbb{R}^n$ open, as a basis.

CH. 2 : SMOOTH MAPS

· <u>DEF. 2.1</u>: Let M be a smooth n-mnfd and let $f: M \to \mathbb{R}^k$ be a fnct, where $k \in \mathbb{N}$. We say that f is a <u>smooth fnct</u> if for every pt $p \in M$ there exists a smooth chart (U, φ) for M s.t. $p \in U$ and $f \circ \varphi^{-1}$ is smooth on the open subset $\varphi(U) \subseteq \mathbb{R}^n$.



· <u>REMARK</u>:

1) If M is a smooth mnfd and $f: M \to \mathbb{R}^k$ is a smooth fnct, then $f \circ \varphi^{-1}: \varphi(U) \to \mathbb{R}^k$ is smooth for every smooth chart (U, φ) for M (see ES3).

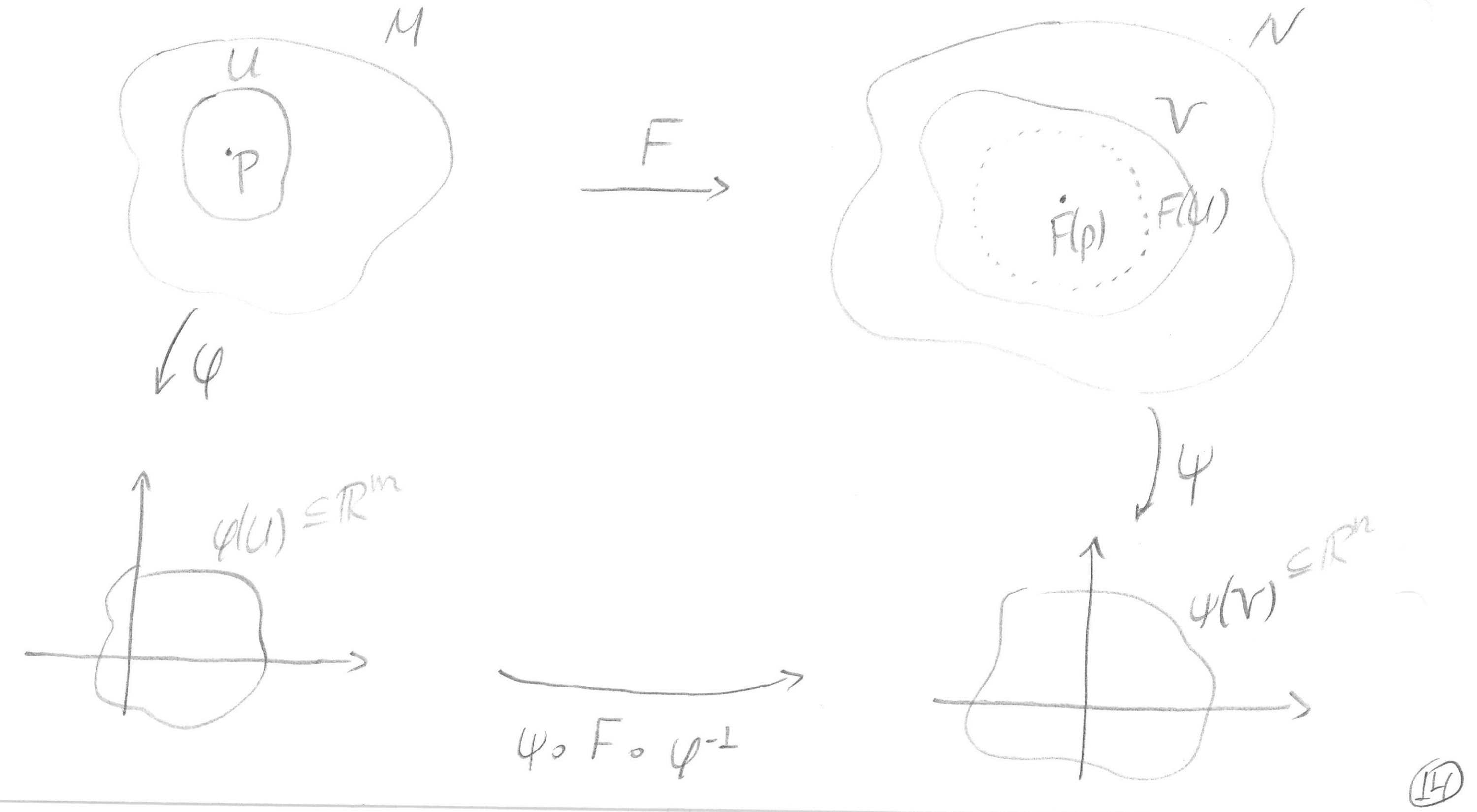
2) Let M be a smooth mnfd. The set $C^\infty(M)$ of all smooth real-valued fncts on M is an infinite-dim $\mathbb{R}$-v.s.: sums and constant multiples of smooth fncts are smooth (see also ES3). Moreover, pointwise multiplication turns

$C^\infty(M)$ into a commutative ring and a commutative and associative $\mathbb{R}$-algebra.

DEF. 2.2: Let M be a smooth mnfd. Given a fnct $f: M \to \mathbb{R}^k$ and a chart (U,φ) for M, the fnct $\hat{f} = f \circ \varphi^{-1}: \varphi(U) \to \mathbb{R}^k$ is called the coordinate representation of f.

By dfn, f is smooth iff its coordinate representation is smooth in some smooth chart around each pt. By the previous REM (1), smooth fncts have smooth coordinate representations in every smooth chart.

DEF. 2.3: Let $F: M \to N$ be a map between smooth mnfds. We say that F is a smooth map if for every $p \in M$ there exist smooth charts (U,φ) containing p and (V,ψ) containing $F(p)$ s.t. $F(U) \subseteq V$ and $\psi \circ F \circ \varphi^{-1}: \varphi(U) \to \psi(V)$ is smooth.



Observe that DEF. 2.1 is a special case of DEF. 2.3 by taking $N = V = \mathbb{R}^k$ and $\psi = Id_{\mathbb{R}^k}$.

· PROP. 2.4: Every smooth map is continuous.

PROOF: Let $F: M \to N$ be a smooth map between smooth mnfds. Fix $p \in M$. Since F is smooth, there are smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ s.t. $F(U) \subseteq V$ and $\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is smooth, and hence cont. Since $F(U) \subseteq V$ and the maps φ and ψ are homeomorphisms, the map

$$F|_U = \psi^{-1} \circ (\psi \circ F \circ \varphi^{-1}) \circ \varphi : U \to V$$

is cont. as a composition of cont. maps. Hence, F is cont. in a neighborhood of each pt, and thus cont. on M. ■

· COMMENT: The requirement that "$\forall p \in M \ \exists (U, \varphi) \ni p \ \exists (V, \psi) \ni F(p)$ s.t. $F(U) \subseteq V$" in the dfn of smoothness is included precisely so that smoothness implies continuity.

· DEF. 2.5: Let $F: M \to N$ be a smooth map between smooth mnfds. If (U, φ) and (V, ψ) are smooth charts for M and N, respectively, then we call $\hat{F} = \psi \circ F \circ \varphi^{-1}$ the coordinate representation of F w.r.t. the given coordinates. It maps $\varphi(U \cap F^{-1}(V))$ to $\psi(V)$.

· REMARK: Let $F: M \to N$ be a smooth map between smooth mnfds. Then the coordinate representation of F w.r.t. every pair of smooth charts for M and N is smooth (see ES3).

- There are equivalent characterizations of smoothness (see ES3). For example, a map $F: M \to N$ between smooth mnfds is smooth iff F is cont. and there exist smooth atlantes $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ for M and N, respectively, s.t. for each α and β, $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is a smooth map from $\varphi_\alpha(U_\alpha \cap F^{-1}(V_\beta))$ to $\psi_\beta(V_\beta)$.

- Smoothness is local (see ES3): if $F: M \to N$ is a map between smooth mnfds and if $\forall p \in M \; \exists p \in U \subseteq M$ s.t. $F|_U$ is smooth, then F is smooth.

- Gluing lemma for smooth maps: Let M and N be smooth mnfds and let $\{U_\alpha\}_{\alpha \in A}$ be an open cover for M. Suppose that for each $\alpha \in A$ we are given a smooth map $F_\alpha: U_\alpha \to N$ s.t. the maps agree on overlaps: $F_\alpha|_{U_\alpha \cap U_\beta} = F_\beta|_{U_\alpha \cap U_\beta}$ for all $\alpha, \beta \in A$. Then there exists a unique smooth map $F: M \to N$ s.t. $F|_{U_\alpha} = F_\alpha$ for each $\alpha \in A$.

· PROP. 2.6: Let M, N and P be smooth mnfds.

(a) Every constant map $c: M \to N$ is smooth.

(b) The identity map Id_M of M is smooth.

(c) If $U \subseteq M$ is an open submanifold, then the inclusion map $\iota: U \hookrightarrow M$ is smooth.