Differential Geometry II - Smooth Manifolds
Winter Term 2023/2024
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Exercise Sheet 2

Exercise 1:
Let M be a topological manifold. Prove the following assertions:

(a) Every smooth atlas $\mathcal{A}$ for M is contained in a unique maximal smooth atlas, called the *smooth structure determined by* $\mathcal{A}$.

[Hint: Consider the collection of all charts that are smoothly compatible with every chart in $\mathcal{A}$.]

(b) Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.

Exercise 2:
Consider the topological manifold $\mathbb{R}$ together with the two atlases $(\mathbb{R}, \mathrm{Id}_{\mathbb{R}})$ and $(\mathbb{R}, \psi)$, where $\psi\colon \mathbb{R} \to \mathbb{R},\ x \mapsto x^3$. Show that the corresponding smooth structures on $\mathbb{R}$ are different, but they are diffeomorphic to each other, i.e., there is a diffeomorphism $(\mathbb{R}, \mathrm{Id}_{\mathbb{R}}) \to (\mathbb{R}, \psi)$.

Exercise 3 (*Finite-dimensional vector spaces*)**:**
Let V be an $\mathbb{R}$-vector space of dimension n. Recall that any norm on V determines a topology, which is independent of the choice of norm. Show that V has a natural smooth manifold structure as follows:

(a) Pick a basis $E_1, \dots, E_n$ for V and consider the map

$$E\colon \mathbb{R}^n \to V,\ (x^1, \dots, x^n) \mapsto \sum_{i=1}^{n} x^i E_i.$$

Show that (V, E^{-1}) is a chart for V; in particular, with the topology defined above, V is thus a topological n-manifold.

(b) Given a different basis $\widetilde{E}_1, \dots, \widetilde{E}_n$ for V, show that the charts (V, E^{-1}) and $(V, \widetilde{E}^{-1})$ are smoothly compatible. The collection of all such charts of V defines a smooth structure, called the *standard smooth structure on* V.

Exercise 4:
Prove the following assertions:

(a) The space $M(m \times n, \mathbb{R})$ of $m \times n$ matrices with real entries has a natural smooth manifold structure.

(b) The *general linear group* $\mathrm{Gl}(n, \mathbb{R})$ (i.e., the group of invertible $n \times n$ matrices with real entries) has a natural smooth manifold structure.

(c) The subset $M_m(m \times n, \mathbb{R})$ of $M(m \times n, \mathbb{R})$ of matrices of rank m, where $m < n$ has a natural smooth manifold structure. Similarly for $M_n(m \times n, \mathbb{R})$ when $n < m$.

(d) The space $\mathcal{L}(V, W)$ of $\mathbb{R}$-linear maps from V to W, where V and W are two finite-dimensional $\mathbb{R}$-vector spaces, has a natural smooth manifold structure.

What is the dimension of each of the above smooth manifolds?

Exercise 5 (*Product manifolds*)**:**
Let $M_1, \ldots, M_k$ be smooth manifolds of dimensions $n_1, \ldots, n_k$, respectively, where $k \geq 2$. Show that the product space $M_1 \times \ldots \times M_k$ is a smooth manifold of dimension $n_1 + \ldots + n_k$ by constructing a smooth manifold structure on it.
In particular, the *n-torus* $\mathbb{T}^n := \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$ is a smooth n-manifold.

Exercise 6 (to be submitted by Friday, 06.10.2023, 20:00):
Consider the n-sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$. Denote by $N = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ the *north pole* and by $S = -N = (0, \ldots, 0, -1)$ the *south pole* of $\mathbb{S}^n$. Define the *stereographic projection from the north pole* N as follows:

$$\sigma \colon \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n, \quad \sigma(x^1, \ldots, x^{n+1}) = \frac{1}{1 - x^{n+1}} \, (x^1, \ldots, x^n).$$

Let $\widetilde{\sigma}(x) = -\sigma(-x)$ for $x \in \mathbb{S}^n \setminus \{S\}$; it is called the *stereographic projection from the south pole.*

(a) For any $x \in \mathbb{S}^n \setminus \{N\}$, show that $\sigma(x) = u$, where $(u, 0)$ is the point where the line through N and x intersects the linear subspace where $x^{n+1} = 0$. Similarly, show that $\widetilde{\sigma}(x)$ is the point where the line through S and x intersects the same subspace.

(b) Show that σ is bijective, and

$$\sigma^{-1}(u^1, \ldots, u^n) = \frac{1}{|u|^2 + 1} \, (2u^1, \ldots, 2u^n, |u|^2 - 1).$$

(c) Verify that the atlas consisting of the two charts $\big(\mathbb{S}^n \setminus \{N\}, \sigma\big)$ and $\big(\mathbb{S}^n \setminus \{S\}, \widetilde{\sigma}\big)$ is a smooth atlas for $\mathbb{S}^n$, and hence defines a smooth structure on $\mathbb{S}^n$. (The coordinates defined by σ or $\widetilde{\sigma}$ are called *stereographic coordinates.*)

(d) Show that the smooth structure determined by the above atlas is the same as the one defined via graph coordinates in the lecture.