Differential Geometry II - Smooth Manifolds
Winter Term 2023/2024
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Exercise Sheet 1 – Solutions

Exercise 1: Show that if a topological space M is locally Euclidean at some point $p \in M$ (i.e., p has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$), then p has a neighborhood that is homeomorphic to the whole space $\mathbb{R}^n$ or to an open ball in $\mathbb{R}^n$.

Solution: We know that there is an open neighborhood U of p and a homeomorphism φ from U to an open subset $\varphi(U)$ of $\mathbb{R}^n$. We can find a ball $B(\varphi(p), r) \subseteq \varphi(U) \subseteq \mathbb{R}^n$ for some $r > 0$. Consider the map $\psi\colon B(\varphi(p), r) \to \mathbb{R}^n$ given by

$$\psi(x) := \frac{x - \varphi(p)}{r - \|x - \varphi(p)\|}.$$

One can easily verify that ψ is a homeomorphism with inverse

$$\psi^{-1}(y) = \varphi(p) + \frac{y}{1 + \|y\|}.$$

Set $U' := \varphi^{-1}(B(\varphi(p), r)) \subseteq M$ and observe that U' is a neighborhood of p in M. Then the map

$$\theta := \psi \circ \varphi|_{U'} : U' \to \mathbb{R}^n$$

is a homeomorphism as both ψ and φ are.

Exercise 2: Examine which of the following spaces (endowed with the subspace topology) is locally Euclidean:

(a) The closed interval $[0, 1] \subseteq \mathbb{R}$.

(b) The "bent line" $\{(x, y) \in \mathbb{R}^2 \mid x \geq 0,\ y \geq 0,\ xy = 0\} \subseteq \mathbb{R}^2$.

Solution:

(a) The interval $[0, 1]$ is *not* locally Euclidean. Suppose by contradiction that it is locally Euclidean. By *Exercise* 1, there is a neighborhood $U \subseteq [0, 1]$ of 0 which is homeomorphic to $\mathbb{R}^n$ for some $n \geq 1$. Denote by $\varphi\colon U \to \mathbb{R}^n$ a homeomorphism and note that U is connected, and thus of the form $U = [0, \varepsilon)$ for some $\varepsilon > 0$. But then $U \setminus \{0\} = (0, \varepsilon)$

is homeomorphic to $\mathbb{R}^n \setminus \{\varphi(0)\}$, and since $(0,\varepsilon)$ is still connected, we infer that $n > 1$ ($\mathbb{R}$ minus a point has two connected components). Now there are two ways to conclude: First, note that $(0,\varepsilon)$ and $\mathbb{R}^n \setminus \{\varphi(0)\}$ are topological manifolds of dimension 1 and n, respectively, and since the dimension of a topological manifold is a topological invariant, we obtain $n = 1$, a contradiction. Second, if $x \in (0,\varepsilon)$, then $(0,\varepsilon)\setminus\{x\}$ is homeomorphic to $\mathbb{R}^n\setminus\{\varphi(0),\varphi(x)\}$; as $n > 1$, the latter is connected, while the former is not, a contradiction.

(b) The "bent line"

$$L \coloneqq \big\{(x,y) \in \mathbb{R}^2 \mid x \geq 0,\ y \geq 0,\ xy = 0\big\}$$

is locally Euclidean. Indeed, denote by $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ the counterclockwise rotation around the origin by 45°. As this is a homeomorphism, we obtain that $L \cong \varphi(L)$. But now note that $\varphi(L)$ coincides with the graph of the absolute value function $|\bullet|\colon \mathbb{R} \to \mathbb{R}$. Thus, we obtain $L \cong \varphi(L) \cong \mathbb{R}$.

Exercise 3: Consider the set

$$X = \big\{(x,y) \in \mathbb{R}^2 \mid y \in \{-1,1\}\big\} \subseteq \mathbb{R}^2$$

and let M be the quotient of X by the equivalence relation generated by $(x,-1) \sim (x,1)$ for all $x \neq 0$. Show that M is locally Euclidean and second-countable, but not Hausdorff.

Solution: Denote by $\pi\colon X \to M$ the quotient map $(x,y) \mapsto [(x,y)]$. The two "origins" are the equivalence classes of the points $(0,y) \in X$ for $y = \pm 1$; these classes have just one element each and we denote them by $0_y = [(0,y)] = \{(0,y)\} \in M$. In contrast, the equivalence class of any other point $(x,y) \in X$ with $x \neq 0$ is the two-point set $\widetilde{x} = [(x,y)] = \{(x,1),(x,-1)\} \in M$. Therefore, M is the set of equivalence classes

$$M = X/\sim = \{0_1\} \cup \{0_{-1}\} \cup \{\widetilde{x}\}_{x\neq 0}.$$

The space M is locally Euclidean of dimension 1 because it is the union of two open sets

$$\mathbb{R}_y = \big\{[(x,y)] \in M \mid x \in \mathbb{R}\big\} \quad \text{(for } y = \pm 1\text{)},$$

each of which is homeomorphic to $\mathbb{R}$ via the map

$$\begin{aligned} \varphi_y\colon \mathbb{R} &\to \mathbb{R}_y \\ x &\mapsto [(x,y)]. \end{aligned}$$

To see that the sets $\mathbb{R}_y$ are open in the quotient topology, note that

$$\pi^{-1}(\mathbb{R}_y) = X \setminus \big\{(0,-y)\big\},$$

which is open in X.

Moreover, M is second-countable because it is the union of two second-countable open subsets, namely, the sets $\mathbb{R}_y \cong \mathbb{R}$ (for $y = \pm 1$).

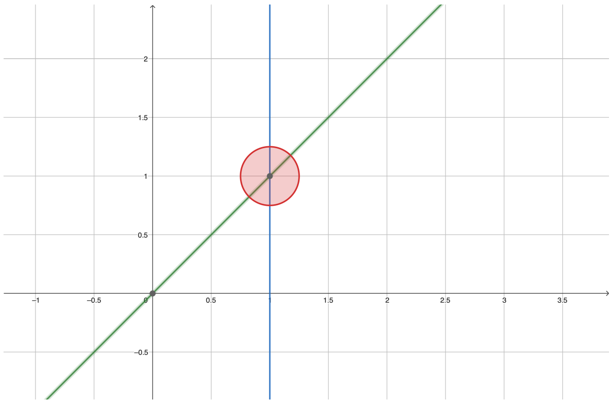
Finally, M is not Hausdorff: let U_{-1} be any open set containing 0_{-1} and let U_1 be any open set containing 0_1. For $y \in \{-1,1\}$, as $\pi^{-1}(U_y)$ is an open subset of X containing $(0,y)$, it contains a set of the form $V_y = (-\varepsilon_y,\varepsilon_y) \times \{y\}$ for some $\varepsilon_y > 0$. Now let x be a real number such that $0 < x < \min\{\varepsilon_{-1},\varepsilon_1\}$. Then $[(x,-1)] = [(x,1)]$ is contained in both U_{-1} and U_1. Hence, 0_{-1} and 0_1 cannot be separated by disjoint open neighborhoods.

Exercise 4: Consider the subset

$$V = \{(x, y) \in \mathbb{R}^2 \mid (x-1)(x-y) = 0\} \subseteq \mathbb{R}^2$$

endowed with the subspace topology. Show that V is not a topological manifold.

Solution: The subset $V \subseteq \mathbb{R}^2$ and a disc with small radius centered at the point $(1, 1) \in \mathbb{R}^2$ (which is the point of intersection of the lines $y = x$ and $x = 1$) have been plotted below.



Since V is a subspace of $\mathbb{R}^2$, it is Hausdorff and second-countable. By considering any point $p \in V \setminus \{(1, 1)\}$, we conclude that if V were a topological manifold, then it would necessarily have dimension 1. Assume now by contradiction that V is a topological 1-manifold. Then there exists an open neighborhood W of $(1, 1)$ which is homeomorphic to an open subset G of $\mathbb{R}$; denote by φ this homeomorphism. For sufficiently small $\varepsilon > 0$, the set $U := B\big((1, 1), \varepsilon\big) \cap W$ (the red disc above) is an open neighborhood of $(1, 1)$ in W, which is connected. Hence, its homeomorphic image $I := \varphi(U)$ in $G \subseteq \mathbb{R}$ is connected as well, and thus $I \subseteq \mathbb{R}$ is an open interval. Observe now that $U \setminus \{(1, 1)\}$ has four connected components, whereas $I \setminus \{\varphi(1, 1)\}$ has only two connected components, a contradiction. In conclusion, V is not a topological manifold.

Exercise 5: Let $M_1, \ldots, M_k$ be topological manifolds of dimensions $n_1, \ldots, n_k$, respectively, where $k \geq 2$. Show that the product space $M_1 \times \ldots \times M_k$ is a topological manifold of dimension $n_1 + \ldots + n_k$.

In particular, the *n-torus* $\mathbb{T}^n := \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$ is a topological n-manifold.

Solution: Any finite product of Hausdorff spaces is also Hausdorff: two distinct points of the product differ at some coordinate, where we can separate them by two disjoint neighborhoods. Moreover, if for each $1 \leq i \leq k$ we denote by $\mathcal{B}_i$ a countable basis for the topology of M_i, then

$$\mathcal{B} := \{B_1 \times \cdots \times B_k \mid \forall 1 \leq i \leq k : \, B_i \in \mathcal{B}_i\}$$

is a countable basis for the topology of the product $M_1 \times \cdots \times M_k$. Finally, given any point $P = (p_1, \ldots, p_k) \in M_1 \times \cdots \times M_k$, by *Exercise 1* we know that for every $1 \leq i \leq k$ there exists an open neighborhood $U_i \subseteq M_i$ of p_i such that $U_i \cong \mathbb{R}^{n_i}$. Therefore, $U := U_1 \times \cdots \times U_k$ is an open neighborhood of P such that $U \cong \mathbb{R}^{n_1 + \ldots + n_k}$. In conclusion, $M_1 \times \cdots \times M_k$ is a topological manifold of dimension $n_1 + \ldots + n_k$.