· <u>REMARK</u>: Let $F: M \to N$ be a smooth map between smooth mnfds. Then the coordinate representation of F w.r.t. every pair of smooth charts for M and N is smooth (see ES3).

- There are equivalent characterizations of smoothness (see ES3). For example, a map $F: M \to N$ between smooth mnfds is smooth iff F is cont. and there exist smooth atlantes $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ for M and N, respectively, s.t. for each α and β, $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is a smooth map from $\varphi_\alpha(U_\alpha \cap F^{-1}(V_\beta))$ to $\psi_\beta(V_\beta)$.

- Smoothness is local (see ES3): if $F: M \to N$ is a map between smooth mnfds and if $\forall p \in M \ \exists p \in U \subseteq M$ s.t. $F|_U$ is smooth, then F is smooth.

L2

L3

- <u>Gluing lemma for smooth maps</u>: Let M and N be smooth mnfds and let $\{U_\alpha\}_{\alpha \in A}$ be an open cover for M. Suppose that for each $\alpha \in A$ we are given a smooth map $F_\alpha: U_\alpha \to N$ s.t. the maps agree on overlaps: $F_\alpha|_{U_\alpha \cap U_\beta} = F_\beta|_{U_\alpha \cap U_\beta}$ for all $\alpha, \beta \in A$. Then there exists a unique smooth map $F: M \to N$ s.t. $F|_{U_\alpha} = F_\alpha$ for each $\alpha \in A$.

· <u>PROP. 2.6</u>: Let M, N and P be smooth mnfds.

(a) Every constant map $c: M \to N$ is smooth.

(b) The identity map Id_M of M is smooth.

(c) If $U \subseteq M$ is an open submanifold, then the inclusion map $\iota: U \hookrightarrow M$ is smooth.

(d) If $F: M \to N$ and $G: N \to P$ are smooth, then so is $G \circ F: M \to P$.

PROOF: ES3. ∎

· EXAMPLE 2.7: Consider the unit n-sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ with its standard smooth structure. The inclusion map $\iota: \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ is continuous (inclusion map of top. subsp.). It is a smooth map, because its coordinate representation w.r.t. any of the graph coordinates of EX. 1.3(2) is

$$\hat{\iota}(u^1, \dots, u^n) = \iota \circ (\varphi_i^{\pm})^{-1}(u^1, \dots, u^n)$$
$$= \left(u^1, \dots, u^{i-1}, \pm\sqrt{1-|u|^2}, u^i, \dots, u^n\right),$$

which is smooth on its domain (the set where $|u|^2 < 1$).

· DEF. 2.8: Let M and N be smooth mnflds.

- A diffeomorphism from M to N is a smooth bijective map $M \to N$ that has a smooth inverse.
- We say that M and N are diffeomorphic if there exists a diffeomorphism between them.

· EXAMPLE 2.9:

1) Consider the maps

$$F: \mathbb{B}^n \to \mathbb{R}^n, \; x \mapsto \frac{x}{\sqrt{1-|x|^2}}$$

and

$$G: \mathbb{R}^n \to \mathbb{B}^n, \; y \mapsto \frac{y}{\sqrt{1+|y|^2}}$$

These maps are smooth, and it is straightforward to check that they are inverses to each other. Thus, they are both diffeomorphisms, so $\mathbb{B}^n \cong \mathbb{R}^n$.

2) If M is a smooth mnfd and if (U,φ) is a smooth chart on M, then $\varphi: U \longrightarrow \varphi(U) \subseteq \mathbb{R}^n$ is a diffeomorphism. (In fact, it has an identity map as a coordinate representation.)

· PROP. 9.10 (Properties of diffeomorphisms): ← Proof = Exercise!

(a) Every composition of diffeo's is a diffeo.

(b) Every finite product of diffeo's between smooth mnfds is a diffeo.

(c) Every diffeo is a homeo and an open map.

(d) The restriction of a diffeo to an open submnfd is a diffeo onto its image.

(e) "Diffeomorphic" is an equivalence relation on the class of all smooth mnfds.

Just as two top. sp. are considered to be "the same" if they are homeomorphic, two smooth mnfds are essentially indistinguishable if they are diffeomorphic. The central concern of smooth mnfd theory is the study of properties of smooth mnfds that are preserved by diffeomorphisms. The dimension is one such property: a non-empty smooth m-mnfd cannot be diffeomorphic to a non-empty smooth n-mnfd unless $m=n$. (This is a consequence of the chain rule.)

Finally, we discuss partitions of unity, which are tools for "blending together" local smooth objects into global ones without necessarily assuming that they agree on overlaps (cf. p.18, gluing lemma). They are indispensable in smooth mnfd theory, and we will see later some first applications of partitions of unity.

• DEF. 2.11: Let M be a top. sp. and let $f: M \to \mathbb{R}^k$ be a fnct. The support of f is defined as

$$\operatorname{supp} f = \overline{\{ p \in M \mid f(p) \neq 0 \}} .$$

- If $\operatorname{supp} f$ is contained in some open subset $U \subseteq M$, we say that f is supported in U.
- If $\operatorname{supp} f$ is a compact set (e.g., if M is a compact space), we say that f is compactly supported.

• DEF. 2.12: Let M be a top. sp. and let $\mathcal{X} = (X_\alpha)_{\alpha \in A}$ be an open cover of M, indexed by a set A. A partition of unity subordinate to $\mathcal{X}$ is an indexed family $(\psi_\alpha)_{\alpha \in A}$ of cont. fncts $\psi_\alpha : M \to \mathbb{R}$ with the following properties:

(i) $0 \leq \psi_\alpha(x) \leq 1$, $\forall \alpha \in A$ $\forall x \in M$.

(ii) $\operatorname{supp} \psi_\alpha \subseteq X_\alpha$, $\forall \alpha \in A$.

(iii) The family of supports $(\operatorname{supp} \psi_\alpha)_{\alpha \in A}$ is locally finite, i.e., every pt has a neighborhood that intersects $\operatorname{supp} \psi_\alpha$ for only finitely many values of α.

(iv) $\sum_{\alpha \in A} \psi_\alpha(x) = 1$, $\forall x \in M$.

Due to the local finiteness condition (iii), the sum in (iv) has only finitely many non-zero terms in a neighborhood of each pt, so there is no issue of convergence.

If M is a smooth mnfd in DEF. 2.12, then a <u>smooth partition of unity</u> is one for which each of the fncts ψ_α is smooth.

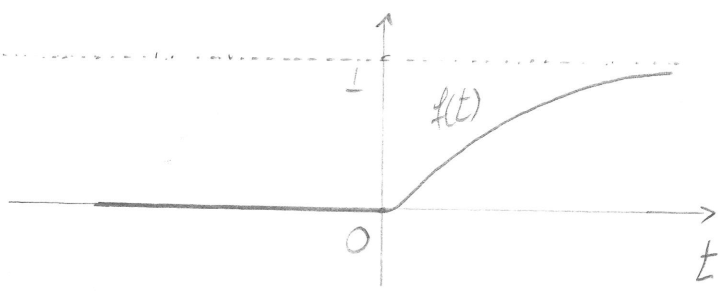
· <u>THM 2.13</u> (Existence of smooth partitions of unity): *Let M be a smooth mnfd and let $\mathcal{X} = (X_\alpha)_{\alpha \in A}$ be an open cover of M. Then there exists a smooth partition of unity subordinate to $\mathcal{X}$.*

For a detailed proof of THM 2.13 we refer to [Lee, Thm 2.23]. We will only review the main ingredients for the proof of THM 2.13:

① <u>Inputs from analysis</u>:

· The function

$$f: \mathbb{R} \longrightarrow \mathbb{R}, \ f(t) = \begin{cases} e^{-\frac{1}{t}} & , t > 0 \\ 0 & , t \leq 0 \end{cases}$$



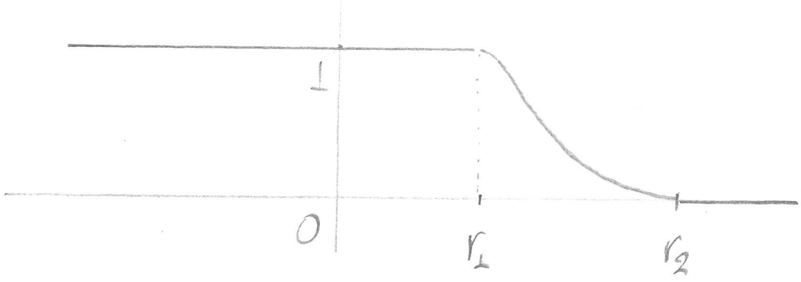
is smooth. [Lee, Lemma 2.21]

· (Existence of cutoff fncts): Given $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$, there exists a smooth fnct $h: \mathbb{R} \longrightarrow \mathbb{R}$ s.t.

$$h(t) \equiv 1 \quad \text{for} \quad t \le r_1$$

$$0 < h(t) < 1 \quad \text{for} \quad r_1 < t < r_2$$

$$h(t) \equiv 0 \quad \text{for} \quad t \ge r_2 .$$



(e.g., take $h(t) := \dfrac{f(r_2-t)}{f(r_2-t)+f(t-r_1)}$, where f is as above)

- Given $r_1, r_2 \in \mathbb{R}$ with $0 < r_1 < r_2$, there exists a smooth fnct $H: \mathbb{R}^n \to \mathbb{R}$ s.t.

$$H \equiv 1 \quad \text{on} \quad \overline{B}_{r_1}(0)$$

$$0 < H(x) < 1 \quad \text{for} \quad x \in B_{r_2}(0) \setminus \overline{B}_{r_1}(0)$$

$$H \equiv 0 \quad \text{on} \quad \mathbb{R}^n \setminus B_{r_2}(0)$$

(e.g., take $H(x) = h(|x|)$, where h is as above).

② Inputs from topology:

"paracompactness" ⇝ one of the main reasons why second countability is included in the dfn of top. mnfds.

- Let M be a top. sp. A collection $\mathcal{X}$ of subsets of M is called locally finite if each pt of M has a neighborhood that intersects at most finitely many of the sets in $\mathcal{X}$.

Given a cover $\mathcal{U}$ of M, another cover $\mathcal{V}$ of M is called a <u>refinement of $\mathcal{U}$</u> if for each $V \in \mathcal{V}$ there exists some $U \in \mathcal{U}$ s.t. $V \subseteq U$. We say that M is <u>paracompact</u> if every open cover of M admits an open, locally finite refinement.

· (Manifolds are paracompact): Every top. mnfd is paracompact. In fact, given a top. mnfd M, an open cover $\mathcal{X}$ of M and any basis $\mathcal{B}$ for the topology of M, there exists a countable, locally finite, open refinement of $\mathcal{X}$ consisting of elements of $\mathcal{B}$. [Lee, Thm 1.15]

Finally, we present some applications of partitions of unity.

① <u>Existence of smooth bump functions</u>:

If M is a top sp., $A \subseteq M$ is a closed subset and $U \subseteq M$ is an open subset s.t. $A \subseteq U$, a cont. fnct $\psi : M \longrightarrow \mathbb{R}$ is called a <u>bump function for A supported in U</u> if

$$0 \leq \psi(x) \leq 1 \ , \ \forall x \in M$$
$$\psi \equiv 1 \text{ on } A$$
$$\operatorname{supp} \psi \subseteq U.$$

· <u>PROP. 2.14</u>: Let M be a smooth mnfd. For any closed subset $A \subseteq M$ and any open subset $U \subseteq M$ containing A, there exists a smooth bump fnct for A supported in U.

<u>PROOF</u>: Set $U_0 := U$ and $U_1 := M \setminus A$, and let $\{\psi_0, \psi_1\}$ be a smooth partition of unity subordinate to the open cover

$\{U_0, U_1\}$ of M. Since $\psi_1 \equiv 0$ on A, and thus $\psi_1 = \sum \psi_i \equiv 1$ on A, the fnct ψ_0 has the required properties. ∎

② Extension lemma for smooth fncts:

Let M and N be smooth mnfds and let $A \subseteq M$ be an arbitrary subset. We say that a map $F: A \longrightarrow N$ is smooth on A if it has a smooth extension in a neighborhood of each pt; namely, if for every $p \in A$ there exists an open subset $p \in W \subseteq M$ and a smooth map $\widetilde{F}: W \longrightarrow N$ whose restriction to $W \cap A$ agrees with F.

· LEM 2.15: Let M be a smooth mnfd, $A \subseteq M$ a closed subset, and $f: A \longrightarrow \mathbb{R}^k$ a smooth fnct. For any open subset $U \subseteq M$ containing A, there exists a smooth fnct $\widetilde{f}: M \longrightarrow \mathbb{R}^k$ s.t. $\widetilde{f}|_A = f$ and $\operatorname{supp} \widetilde{f} \subseteq U$.

PROOF: For each $p \in A$, choose a neighborhood W_p of p and a smooth fnct $\widetilde{f}_p: W_p \longrightarrow \mathbb{R}^k$ s.t. $\widetilde{f}_p|_{W_p \cap A} = f$. Replacing W_p by $W_p \cap U$, w.m.a.t. $W_p \subseteq U$. The family of sets $\{W_p\}_{p \in A} \cup \{M \setminus A\}$ is an open cover of M. Let $\{\psi_p\}_{p \in A} \cup \{\psi_0\}$ be a smooth partition of unity subordinate to this cover, with $\operatorname{supp} \psi_p \subseteq W_p$ and $\operatorname{supp} \psi_0 \subseteq M \setminus A$.

For each $p \in A$, the product $\psi_p \widetilde{f}_p$ is smooth on W_p, and has a smooth extension to all of M if we interpret it to be zero on $M \setminus \operatorname{supp} \psi_p$. Thus, we can define the fnct

$$\widetilde{f}: M \longrightarrow \mathbb{R}^k, \; x \mapsto \sum_{p \in A} \psi_p(x) \widetilde{f}_p(x).$$

Since the collection of supports $\{\operatorname{supp}\psi_p\}_{p\in A}$ is locally finite, this sum actually has only finitely many nonzero terms in a neighborhood of any pt of M, and therefore defines a smooth fnct. If $x \in A$, then $\psi_0(x) = 0$ and $\tilde{f}_p(x) = f(x)$ for each p s.t. $\psi_p(x) \neq 0$, so

$$\tilde{f}(x) = \sum_{p\in A} \psi_p(x)\tilde{f}_p(x) = \Big(\psi_0(x) + \sum_{p\in A}\psi_p(x)\Big) f(x) = f(x).$$

Thus, $\tilde{f}$ is indeed an extension of f. Finally, we have

$$\operatorname{supp}\tilde{f} \subseteq \overline{\bigcup_{p\in A}\operatorname{supp}\psi_p} = \bigcup_{p\in A}\operatorname{supp}\psi_a \subseteq U. \quad \blacksquare$$

↳ property of locally finite collections; see [Lee, Lemma 1.13]

COMMENT:

1) The conclusion of the extension lemma can be false if A is not closed.

2) The assumption in the extension lemma that the co-domain is $\mathbb{R}^k$, and not some other mnfd, is necessary (for other codomains, extensions can fail to exist for topological reasons).

③ Existence of smooth exhaustion fncts: [Lee, Prop. 2.28]

④ $\underbrace{[\text{Lee, Thm } 2.29]}_{}^{= \text{THM } 2.16}$: Let M be a smooth mnfd. If K is a closed subset of M, then there exists a smooth non-negative fnct $f: M \to \mathbb{R}$ s.t. $f^{-1}(0) = K$.