Differential Geometry II - Smooth Manifolds
Winter Term 2023/2024
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Exercise Sheet 4

Exercise 1 (**to be submitted by Friday, 20.10.2023, 20:00**):
Let M, N and P be smooth manifolds, let $F\colon M \to N$ and $G\colon N \to P$ be smooth maps, and let $p \in M$. Prove the following assertions:

(a) The map $dF_p\colon T_pM \to T_{F(p)}N$ is $\mathbb{R}$-linear.

(b) $d(G \circ F)_p = dG_{F(p)} \circ dF_p\colon T_pM \to T_{(G\circ F)(p)}P$.

(c) $d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM}\colon T_pM \to T_pM$.

(d) If F is a diffeomorphism, then $dF_p\colon T_pM \to T_{F(p)}N$ is an isomorphism, and it holds that $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

Exercise 2 (*The tangent space to a vector space*)**:**
Let V be a finite-dimensional $\mathbb{R}$-vector space with its standard smooth manifold structure, see *Exercise* 3, *Sheet* 2. Fix a point $a \in V$.

(a) For each $v \in V$ define a map

$$D_v\big|_a\colon C^\infty(V) \longrightarrow \mathbb{R},\ f \mapsto \frac{d}{dt}\bigg|_{t=0} f(a+tv).$$

Show that $D_v\big|_a$ is a derivation at a.

(b) Show that the map

$$V \to T_aV,\ v \mapsto D_v\big|_a$$

is a canonical isomorphism, such that for any linear map $L\colon V \to W$ the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\ \cong\ } & T_aV \\ {\scriptstyle L}\big\downarrow & & \big\downarrow{\scriptstyle dL_a} \\ W & \xrightarrow{\ \cong\ } & T_{La}W. \end{array}$$

Exercise 3 (*The tangent space to a product manifold*):
Let $M_1, \dots, M_k$ be smooth manifolds, where $k \geq 2$. For each $j \in \{1, \dots, k\}$, let

$$\pi_j \colon M_1 \times \ldots \times M_k \to M_j$$

be the projection onto the j-th factor M_j. Show that for any point $p = (p_1, \dots, p_k) \in M_1 \times \ldots \times M_k$, the map

$$\begin{aligned} \alpha \colon T_p\big(M_1 \times \ldots \times M_k\big) &\longrightarrow T_{p_1}M_1 \oplus \ldots \oplus T_{p_k}M_k \\ v &\mapsto \big(d(\pi_1)_p(v), \dots, d(\pi_k)_p(v)\big) \end{aligned}$$

is an $\mathbb{R}$-linear isomorphism.

Exercise 4 (*Tangent vectors as derivations of the space of germs*):
Let M be a smooth manifold and let p be a point of M.

(a) Consider the set $\mathcal{S}$ of ordered pairs (U, f), where U is an open subset of M containing p and $f \colon U \to \mathbb{R}$ is a smooth function. Define on $\mathcal{S}$ the following relation:

$$(U, f) \sim (V, g) \quad \text{if } f \equiv g \text{ on some open neighborhood of } p.$$

Show that $\sim$ is an equivalence relation on $\mathcal{S}$. The equivalence class of an ordered pair (U, f) is denoted by $[(U, f)]$ or simply by $[f]_p$ and is called *the germ of f at p*.

(b) The set of all germs of smooth functions at p is denoted by $C_p^\infty(M)$. Show that $C_p^\infty(M)$ is an $\mathbb{R}$-vector space and an associative $\mathbb{R}$-algebra under the operations

$$\begin{aligned} c[(U, f)] &= [(U, cf)], \text{ where } c \in \mathbb{R}, \\ [(U, f)] + [(V, g)] &= [(U \cap V, f + g)], \\ [(U, f)][(V, g)] &= [(U \cap V, fg)]. \end{aligned}$$

(c) A *derivation* of $C_p^\infty(M)$ is an $\mathbb{R}$-linear map $v \colon C_p^\infty(M) \to \mathbb{R}$ satisfying the following product rule:

$$v[fg]_p = f(p)v[g]_p + g(p)v[f]_p.$$

The set of derivations of $C_p^\infty(M)$ is denoted by $\mathcal{D}_pM$.

(i) Show that $\mathcal{D}_pM$ is an $\mathbb{R}$-vector space.

(ii) Show that the map

$$\Phi \colon \mathcal{D}_pM \to T_pM, \; \Phi(v)(f) = v[f]_p$$

is an isomorphism.

Definition.

(a) Let M be a smooth manifold. A *smooth (parametrized) curve* in M is a smooth map $\gamma \colon J \to M$, where $J \subseteq \mathbb{R}$ is an interval.

(b) Given a smooth manifold M, a smooth curve $\gamma\colon J \to M$ in M and an instant $t_0 \in J$, the *velocity of* γ *at* t_0 is defined to be the tangent vector

$$\gamma'(t_0) \coloneqq d\gamma\left(\left.\frac{d}{dt}\right|_{t_0}\right) \in T_{\gamma(t_0)}M,$$

where $d/dt|_{t_0}$ is the standard coordinate basis vector in $T_{t_0}\mathbb{R}$.

Remark. Assume that M, γ and t_0 are as above. The tangent vector $\gamma'(t_0)$ acts on functions $f \in C^\infty(M)$ by

$$\gamma'(t_0)f = d\gamma\left(\left.\frac{d}{dt}\right|_{t_0}\right)f = \left.\frac{d}{dt}\right|_{t_0}(f \circ \gamma) = (f \circ \gamma)'(t_0).$$

In other words, $\gamma'(t_0)$ is the derivation at $\gamma(t_0)$ obtained by taking the derivative of a function along γ. (If t_0 is an endpoint of the interval $J \subseteq \mathbb{R}$, this still holds, provided that we interpret the derivative with respect to t as a one-sided derivative, or equivalently as the derivative of any smooth extension of $f \circ \gamma$ to an open subset of $\mathbb{R}$.)

Now, let (U, φ) be a smooth chart for M with coordinate functions (x^i). If $\gamma(t_0) \in U$, then we can write the coordinate representation of γ as

$$\gamma(t) = \big(\gamma^1(t), \dots, \gamma^n(t)\big),$$

at least for $t \in J$ sufficiently close to $t_0 \in J$, and then the coordinate formula for the differential yields

$$\gamma'(t_0) = \frac{d\gamma^i}{dt}(t_0)\left.\frac{\partial}{\partial x^i}\right|_{\gamma(t_0)}.$$

This means that $\gamma'(t_0)$ is given by essentially the same formula as it would be in Euclidean space: it is the tangent vector whose components in a coordinate basis are the derivatives of the component functions of γ.

Exercise 5:
Prove the following assertions:

(a) *Tangent vectors as velocity vectors of smooth curves*: Let M be a smooth manifold. If $p \in M$, then for any $v \in T_pM$ there exists a smooth curve $\gamma\colon (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$.

(b) *The velocity of a composite curve*: If $F\colon M \to N$ is a smooth map and if $\gamma\colon J \to M$ is a smooth curve, then for any $t_0 \in J$, the velocity at $t = t_0$ of the composite curve $F \circ \gamma\colon J \to N$ is given by

$$(F \circ \gamma)'(t_0) = dF\big(\gamma'(t_0)\big).$$

(c) *Computing the differential using a velocity vector*: If $F\colon M \to N$ is a smooth map, $p \in M$ and $v \in T_pM$, then

$$dF_p(v) = (F \circ \gamma)'(0)$$

for any smooth curve $\gamma\colon J \to M$ such that $0 \in J$, $\gamma(0) = p$ and $\gamma'(0) = v$.