Differential Geometry II - Smooth Manifolds
Winter Term 2023/2024
Lecturer: Dr. N. Tsakanikas
Assistant: L. E. Rösler

Exercise Sheet 5

Exercise 1:

(a) Let (x, y) denote the standard coordinates on $\mathbb{R}^2$. Verify that $(\widetilde{x}, \widetilde{y})$ are smooth global coordinates on $\mathbb{R}^2$, where

$$\widetilde{x} = x \quad \text{and} \quad \widetilde{y} = y + x^3.$$

Let p be the point $(1, 0) \in \mathbb{R}^2$ (in standard coordinates), and show that

$$\left.\frac{\partial}{\partial x}\right|_p \neq \left.\frac{\partial}{\partial \widetilde{x}}\right|_p,$$

even though the coordinate functions x and $\widetilde{x}$ are identically equal.

(This shows that each coordinate vector $\partial/\partial x^i|_p$ depends on the entire coordinate system, not just on the single coordinate function x^i.)

(b) *Polar coordinates on* $\mathbb{R}^2$: Consider the map

$$\begin{aligned}\Phi \colon W := (0, +\infty) \times (-\pi, \pi) &\to \mathbb{R}^2 \\ (r, \theta) &\mapsto (r\cos\theta,\ r\sin\theta).\end{aligned}$$

(i) Show that Φ is a diffeomorphism onto its image $U := \Phi(W)$.
(Therefore, Φ^{-1} can be considered as a smooth chart on $\mathbb{R}^2$, and it is common to call its component functions the *polar coordinates* (r, θ) on $\mathbb{R}^2$.)

(ii) Let p be a point in $\mathbb{R}^2$ whose polar coordinate representation is $(r, \theta) = (2, \pi/2)$, and let $v \in T_p\mathbb{R}^2$ be the tangent vector whose polar coordinate representation is

$$v = 3\left.\frac{\partial}{\partial r}\right|_p - \left.\frac{\partial}{\partial \theta}\right|_p.$$

Compute the coordinate representation of v in terms of the standard coordinate vectors

$$\left.\frac{\partial}{\partial x}\right|_p, \left.\frac{\partial}{\partial y}\right|_p.$$

(c) *Spherical coordinates on* $\mathbb{R}^3$: Consider the map

$$\begin{aligned}\Psi\colon W := (0,+\infty)\times(0,2\pi)\times(0,\pi) &\to \mathbb{R}^3\\ (r,\varphi,\theta) &\mapsto (r\cos\varphi\sin\theta,\ r\sin\varphi\sin\theta,\ r\cos\theta).\end{aligned}$$

(i) Show that Ψ is a diffeomorphism onto its image $U := \Psi(W)$.
(Therefore, Ψ^{-1} can be considered as a smooth chart on $\mathbb{R}^3$, and it is common to call its component functions the *spherical coordinates* (r,φ,θ) on $\mathbb{R}^3$.)

(ii) Express the coordinate vectors

$$\left.\frac{\partial}{\partial r}\right|_p,\ \left.\frac{\partial}{\partial \varphi}\right|_p,\ \left.\frac{\partial}{\partial \theta}\right|_p$$

of this chart at some point $p\in U$ in terms of the standard coordinate vectors

$$\left.\frac{\partial}{\partial x}\right|_p,\ \left.\frac{\partial}{\partial y}\right|_p,\ \left.\frac{\partial}{\partial z}\right|_p.$$

Exercise 2 (to be submitted by Friday, 27.10.2023, 20:00):
Consider the inclusion $\iota\colon \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$, where both $\mathbb{S}^2$ and $\mathbb{R}^3$ are endowed with the standard smooth structure. Let $p=(p^1,p^2,p^3)\in\mathbb{S}^2$ with $p^3>0$. What is the image of the differential $d\iota_p\colon T_p\mathbb{S}^2\to T_p\mathbb{R}^3$?

Exercise 3:
Let $M_1,\dots,M_k$ be smooth manifolds, where $k\geq 2$. Show that $T(M_1\times\ldots\times M_k)$ is diffeomorphic to $T(M_1)\times\ldots\times T(M_k)$.

Exercise 4:

(a) Let $F\colon M\to N$ be a smooth map. Show that its *global differential* $dF\colon TM\to TN$ (which is just the map whose restriction to each tangent space $T_pM\subseteq TM$ is dF_p) is also a smooth map.

(b) Let $F\colon M\to N$ and $G\colon N\to P$ be smooth maps. Prove the following assertions:

(i) $d(G\circ F) = dG\circ dF\colon TM\to TP$.

(ii) $d(\mathrm{Id}_M)=\mathrm{Id}_{TM}\colon TM\to TM$.

(iii) If F is a diffeomorphism, then $dF\colon TM\to TN$ is also a diffeomorphism, and it holds that $(dF)^{-1}=d(F^{-1})$.

Exercise 5:

(a) Let $f\colon X\to S$ be a map from a topological space X to a set S. Show that if X is connected and if f is *locally constant*, i.e., for every $x\in X$ there exists a neighborhood U of x in X such that $f|_U\colon U\to S$ is constant, then f is constant.

[Hint: Show that f is continuous when S is endowed with the discrete topology.]

(b) Let M and N be smooth manifolds and let $F\colon M \to N$ be a smooth map. Assume that M is connected. Show that $dF_p\colon T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$ if and only if F is constant.

[Hint: Use (a). You may also use (without proof) the fact that any topological manifold is locally (path) connected.]