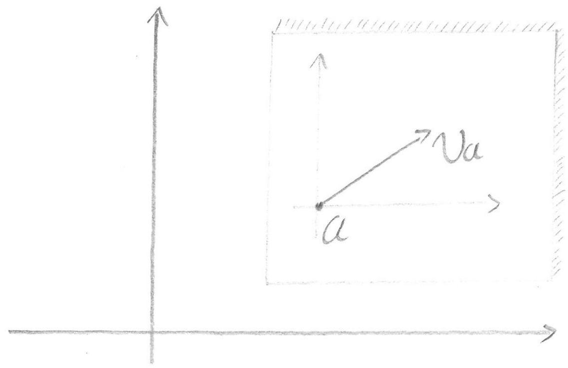
CH. 3 : THE TANGENT BUNDLE

Given a pt $a \in \mathbb{R}^n$, we define <u>the geometric tangent space to $\mathbb{R}^n$ at a</u> to be the set

$$\mathbb{R}^n_a := \{a\} \times \mathbb{R}^n = \{ (a,v) \mid v \in \mathbb{R}^n \} .$$

We abbreviate (a,v) as v_a or $v|_a$, and we think of v_a as the vector v with its initial pt at a.



The set $\mathbb{R}^n_a$ is an $\mathbb{R}$-v.s. under the natural operation

$$v_a + w_a := (v+w)_a ,$$
$$\lambda v_a = (\lambda v)_a ,$$

and the vectors $e_i|_a$, $1 \leq i \leq n$ (where e_i denotes the i-th standard basis vector) are a basis of $\mathbb{R}^n_a$. In fact, $\mathbb{R}^n_a$ is essentially the same as $\mathbb{R}^n$ itself; the only reason we add the index a is so that the geometric tangent spaces $\mathbb{R}^n_a$ and $\mathbb{R}^n_b$ at distinct pts a and b are disjoint sets.

· <u>DEF. 3.1</u> : Given $a \in \mathbb{R}^n$, a map $w : C^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R}$ is called a <u>derivation at a</u> if it is $\mathbb{R}$-linear and satisfies the product rule:

$$w(fg) = f(a)\, w(g) + g(a)\, w(f) .$$

We denote by $T_a\mathbb{R}^n$ the set of all derivations of $C^\infty(\mathbb{R}^n)$ at a. Clearly, $T_a\mathbb{R}^n$ is an $\mathbb{R}$-v.s. under the operations

$$(w_1 + w_2)f = w_1 f + w_2 f,$$
$$(\lambda w)f = \lambda w f.$$

The most important fact about $T_a\mathbb{R}^n$ is that it is fin. dim; in fact, it is naturally isomorphic to the geometric tangent space $\mathbb{R}^n_a$ that we defined above. The proof will be based on the following lemma.

· <u>LEM 3.2</u>: Let $a \in \mathbb{R}^n$, $w \in T_a\mathbb{R}^n$, and $f, g \in C^\infty(\mathbb{R}^n)$.

(a) If f is constant, then $wf = 0$.

(b) If $f(p) = g(p) = 0$, then $w(fg) = 0$.

<u>PROOF</u>:

(a) Consider the constant fnct $f_1 \equiv 1 \in C^\infty(\mathbb{R}^n)$. We have

$$w(f_1) = w(f_1 \cdot f_1) = \overset{1}{\cancel{f_1(a)}}\, w f_1 + \overset{1}{\cancel{f_1(a)}}\, w f_1 \Rightarrow$$
$$\Rightarrow w f_1 = 0.$$

Since $f \equiv c$ is constant, we obtain

$$wf = w(c f_1) = c\, w f_1 = 0.$$

(b) Follows immediately from the product rule.

· <u>PROP. 3.3</u>: Let $a \in \mathbb{R}^n$.

(a) For each geometric tangent vector $v_a \in \mathbb{R}^n_a$, the map

$$D_v|_a : C^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R}$$
$$f \longmapsto D_v|_a f = D_v f(a) = \left.\frac{d}{dt}\right|_{t=0} f(a+tv)$$

(directional derivative of f in the direction v at a) is a derivation of $C^\infty(\mathbb{R}^n)$ at a.

(b) The map

$$\underline{\Phi} : \mathbb{R}^n_a \longrightarrow T_a\mathbb{R}^n$$
$$v \longmapsto D_v\big|_a$$

is an $\mathbb{R}$-linear isomorphism.

(c) The n derivations

$$\frac{\partial}{\partial x^1}\Big|_a, \dots, \frac{\partial}{\partial x^n}\Big|_a$$

defined by

$$\frac{\partial}{\partial x^i}\Big|_a f := \frac{\partial f}{\partial x^i}(a), \quad 1 \le i \le n,$$

form a basis of $T_a\mathbb{R}^n$, and thus

$$\dim_{\mathbb{R}} T_a\mathbb{R}^n = n.$$

<u>PROOF</u>:

(a) Easy to check (using calculus).

(b) · linearity: For every $f \in C^\infty(\mathbb{R}^n)$ we have

$$\underline{\Phi}(\lambda_1 v_1 + \lambda_2 v_2)(f) = D_{\lambda_1 v_1 + \lambda_2 v_2}\big|_a f$$
$$= \frac{d}{dt}\Big|_{t=0} f(a + t(\lambda_1 v_1 + \lambda_2 v_2))$$
$$= Df(a) \cdot (\lambda_1 v_1 + \lambda_2 v_2)$$
$$= \lambda_1 \frac{d}{dt}\Big|_{t=0} f(a + tv_1) + \lambda_2 \frac{d}{dt}\Big|_{t=0} f(a+tv_2)$$
$$= \lambda_1 \underline{\Phi}(v_1)(f) + \lambda_2 \underline{\Phi}(v_2)(f)$$

$$= \left(\lambda_1 \Phi(v_1) + \lambda_2 \Phi(v_2)\right)(f),$$

which shows the $\mathbb{R}$-linearity of Φ.

- injectivity: Suppose that $\Phi(v_a) = D_v|_a = 0$ is the zero derivation. Writing $v_a = v^i e_i|_a$ in terms of the standard basis, and considering the j-th coordinate fnct $x^j : \mathbb{R}^n \to \mathbb{R}$, though of as a smooth fnct on $\mathbb{R}^n$, we obtain

$$0 = D_v|_a x^j = v^i \frac{\partial}{\partial x^i}(x^j)\Big|_{x=a} = v^j,$$

where the last equality follows because $\frac{\partial x^j}{\partial x^i} = 0$, $i \neq j$, and $\frac{\partial x^i}{\partial x^i} = 1$. Hence, $v_a = 0 \in \mathbb{R}^n_a$.

- surjectivity: Let $w \in T_a\mathbb{R}^n$ set $v := v^i e_i|_a$, where $v^i = w(x^i) \in \mathbb{R}$. W.w.s.t. $w = \Phi(v) = D_v|_a$. To this end, let $f \in C^\infty(\mathbb{R}^n)$. By Taylor's thm we can write

$$f(x) = f(a) + \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a)(x^i - a^i) +$$

$$+ \sum_{i,j=1}^n (x^i - a^i)\underbrace{(x^j - a^j)\int_0^1 (1-t)\frac{\partial^2 f}{\partial x^i \partial x^j}(a + t(x-a))\,dt}$$

Note that each term in the above sum is a product of two smooth fncts of x that vanish at $x=a$: one is $(x^i - a^i)$ and the other is $(x^j - a^j)\cdot(\text{integral})$. By LEM 3.2(b) the derivation w annihilates the entire sum. Thus,

$$wf = \overset{\nearrow 0}{w(f(a))} + \sum_{i=1}^n w\left(\frac{\partial f}{\partial x^i}(a)(x^i - a^i)\right)$$

$$= \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a)\left(\overset{\nearrow v^i}{w(x^i)} - \overset{\nearrow 0}{w(a^i)}\right)$$

$$= \sum_{i=1}^{n} v^i \frac{\partial f}{\partial x^i}(a) = D_v|_a(f).$$

(c) We have already used above that if $v = v^i e_i|_a$ (in terms of the standard basis), then $D_v|_a f = v^i \frac{\partial f}{\partial x^i}(a)$ by the chain rule. In particular, if $v = e_i|_a$, then $D_{e_i}|_a f = = \frac{\partial f}{\partial x^i}(a)$ (the i-th derivation defined above), i.e., $\frac{\partial}{\partial x^i}\Big|_a = D_{e_i}|_a$.
Hence, (c) follows from (b). ∎

· DEF. 3.4: Let M be a smooth mnfd and let $p \in M$. A map $v: C^\infty(M) \longrightarrow \mathbb{R}$ is called a derivation at p if it is $\mathbb{R}$-linear and satisfies the product rule:

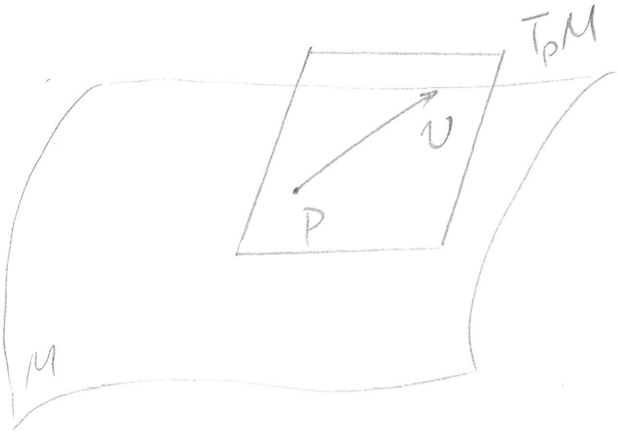
$$v(fg) = f(p)\, vg + g(p)\, vf, \quad \forall f, g \in C^\infty(M).$$

We denote by T_pM the set of all derivations of $C^\infty(M)$ at p. Clearly, T_pM is an $\mathbb{R}$-v.s, called the tangent space to M at $p \in M$. An element of T_pM is called a tangent vector at p.

· LEM 3.5: Let M be a smooth mnfd, $p \in M$, $v \in T_pM$ and $f, g \in C^\infty(M)$.
(a) If f is constant, then $vf = 0$.
(b) If $f(p) = g(p) = 0$, then $v(fg) = 0$.

PROOF: Exercise! (cf. Lemma 3.2) ∎

With the motivation of geometric tangent vectors in $\mathbb{R}^n$ in mind, we visualize tangent vectors to M as "arrows" that are tangent to M and whose base points are attached to M at the given pt.



For alternative descriptions of tangent vectors to M, see ES4 E4.

· <u>DEF. 3.6</u>: If $F: M \to N$ is a smooth map, then for each $p \in M$ we define a map $dF_p : T_pM \to T_{F(p)}N$, called <u>the differential of F at p</u>, as follows. Given $v \in T_pM$, we let $dF_p(v)$ be the derivation at $F(p)$ that acts on $f \in C^\infty(N)$ by

$$dF_p(v)(f) = v(f \circ F).$$

The operator $dF_p(v) : C^\infty(N) \to \mathbb{R}$ is a derivation at $F(p)$: it is $\mathbb{R}$-linear, since v is so, and satisfies the product rule

$$\begin{aligned} dF_p(v)(fg) &= v(fg \circ F) = v((f \circ F)(g \circ F)) \\ &= (f \circ F)(p)\, v(g \circ F) + (g \circ F)(p)\, v(f \circ F) \\ &= f(F(p))\, dF_p(v)(g) + g(F(p))\, dF_p(v)(f). \end{aligned}$$

· <u>PROP. 3.7</u>: Let $F: M \to N$ and $G: N \to P$ be smooth maps and let $p \in M$.

(a) $dF_p : T_pM \longrightarrow T_{F(p)}N$ is an $\mathbb{R}$-linear map.

(b) $d(G\circ F)_p = dG_{F(p)} \circ dF_p : T_pM \longrightarrow T_{(G\circ F)(p)}P$.

(c) $d(Id_M)_p = Id_{T_pM} : T_pM \longrightarrow T_pM$.

(d) If F is a diffeomorphism, then $dF_p : T_pM \longrightarrow T_{F(p)}N$ is an isomorphism, and it holds that $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

PROOF: ES4. ∎

Our first important application of the differential will be to use coordinate charts to relate the tangent space to a pt on a mnfd with the Euclidean tangent space. But there is an important technical issue that we must address first: while the tangent space is defined in terms of smooth fncts on the whole mnfd, coordinate charts are in general defined only on open subsets. The key point, expressed in the next proposition, is that tangent vectors act locally.

· PROP. 3.8: Let M be a smooth mnfd, $p\in M$ and $v\in T_pM$. If $f,g\in C^\infty(M)$ agree on some neighborhood of p, then $vf = vg$.

· PROOF: Set $h := f-g$ and observe that h is a smooth fnct on M that vanishes in a neighborhood of p. By PROP. 2.14 there exists a smooth bump fnct ψ for $\operatorname{supp} h$ supported in $M\setminus\{p\}$ (open in M which contains $\operatorname{supp} h$, since $h(p)=0$). Since $\psi\equiv 1$ where h is non-zero, the product ψh is identically equal to h. Since $h(p) = \psi(p) = 0$, by LEM 3.5(b) we obtain $v(h) = v(\psi h) = 0$, so $v(f) = v(g)$ by linearity. ∎

Using PROP. 3.8, we can identify the tangent space to an open submanifold with the tangent space to the whole mnfd.

· **PROP. 3.9**: Let M be a smooth mnfd, let $U \subseteq M$ be an open subset and let $\iota : U \hookrightarrow M$ be the inclusion map. For every $p \in U$, the differential $d\iota_p : T_pU \longrightarrow T_pM$ is an isomorphism.

PROOF: We first prove injectivity. Let $v \in T_pU$ s.t. $d\iota_p(v) = 0 \in T_pM$. Let V be a neighborhood of p s.t. $\overline{V} \subseteq U$. If $f \in C^\infty(U)$ is arbitrary, then by LEM 2.15 there exists $\tilde{f} \in C^\infty(M)$ s.t. $\tilde{f}|_{\overline{V}} = f$. Since then f and $\tilde{f}|_U$ are smooth fncts that agree in a neighborhood of p, PROP. 3.8 implies

$$vf = v(\tilde{f}|_U) = v(\tilde{f} \circ \iota) = d\iota_p(v)(\tilde{f}) = 0 .$$

Hence, $v = 0 \in T_pU$, so $d\iota_p$ is injective.

L_4 L_5

We now prove surjectivity. Let $w \in T_pM$. Define

$$\begin{aligned} v : C^\infty(U) &\longrightarrow \mathbb{R} \\ f &\longmapsto w\tilde{f} \end{aligned}$$

where $\tilde{f}$ is any smooth fnct on M that agrees with f on $\overline{V}$ (see LEM 2.15). By PROP. 3.8, vf is independent of the choice of $\tilde{f}$, so v is well-defined, and it is easy to check that it is a derivation of $C^\infty(U)$ at p. For any $g \in C^\infty(M)$, we have

$$d\iota_p(v)(g) = v(g \circ \iota) = w(\widetilde{g \circ \iota}) = wg ,$$

where the last equality follows from the fact that $g \circ \iota$, $\widetilde{g \circ \iota}$ and g all agree on V.

■

Given an open subset $U \subseteq M$, the isomorphism $d\iota_p$ between T_pU and T_pM is canonically defined, independent of any choices. From now on we identify T_pU with T_pM for any $p\in U$. This identification just amounts to the observation that $d\iota_p(v)$ is the same derivation as v, thought of as acting on fncts on the bigger mnfd M instead of on fncts on U. Since the action of a derivation on a fnct depends only on the values of the fnct in an arbitrarily small neighborhood, this is a harmless identification. In particular, this means that any tangent vector $v\in T_pM$ can be unambiguously applied to fncts defined only in a neighborhood of p, not necessarily on all of M.

· **PROP. 3.10**: If M is a smooth n-mnfd, then for each $p\in M$, the tangent space T_pM is an n-dim $\mathbb{R}$-v.s.

PROOF: Fix $p\in M$ and let (U,φ) be a smooth coordinate chart containing p. Since $\varphi: U \longrightarrow \hat{U} \subseteq \mathbb{R}^n$ is a diffeomorphism, $d\varphi_p: T_pU \longrightarrow T_{\varphi(p)}\hat{U}$ is an isomorphism by PROP. 3.7(d). Since PROP. 3.9 guarantees that $T_pU \cong T_pM$ and $T_{\varphi(p)}\hat{U} \cong T_{\varphi(p)}\mathbb{R}^n$, it follows from PROP. 3.3(c) that $\dim T_pM = \dim T_{\varphi(p)}\mathbb{R}^n = n$. ■

⇝ the tangent space to a vector space : ES4

⇝ the tangent space to a product mnfd : ES4

Next, we will show how to do computations with tangent vectors and differentials in local coordinates.

Let M be a smooth mnfd and let (U,φ) be a smooth coordinate chart on M. Then φ is a diffeomorphism from U to an open subset $\hat{U} \subseteq \mathbb{R}^n$. By PROPs 3.7(d) and 3.9 we infer that $d\varphi_p: T_pM \longrightarrow T_{\varphi(p)}\mathbb{R}^n$ is an isomorphism (for each $p\in U$).