Differential Geometry II - Smooth Manifolds
Winter Term 2023/2024
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Exercise Sheet 3 – Solutions

Exercise 1: Let M and N be smooth manifolds and let $F\colon M \to N$ be a map. Show that F is smooth if and only if either of the following conditions is satisfied:

(a) For every $p \in M$ there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $U \cap F^{-1}(V)$ is open in M and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi\big(U \cap F^{-1}(V)\big)$ to $\psi(V)$.

(b) F is continuous and there exist smooth atlases $\big\{(U_\alpha, \varphi_\alpha)\big\}$ and $\big\{(V_\beta, \psi_\beta)\big\}$ for M and N, respectively, such that for each α and β, $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is a smooth map from $\varphi_\alpha\big(U_\alpha \cap F^{-1}(V_\beta)\big)$ to $\psi_\beta(V_\beta)$.

Solution:

(a) We prove the two directions:

($\Rightarrow$) Suppose F is smooth and let $p \in M$. Then there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subseteq V$ and such that $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\psi(V)$. Then $U \cap F^{-1}(V) = U$, and thus the charts (U, φ) and (V, ψ) satisfy the conditions specified in (a).

($\Leftarrow$) Assume that (a) holds and let $p \in M$. Let (U, φ) resp. (V, ψ) be the charts given by (a). Then, if we put $U' \coloneqq U \cap F^{-1}(V)$ and $\varphi' \coloneqq \varphi|_{U'}$, we infer that (U', φ') is a smooth chart containing p such that $F(U') \subseteq V$ and such that $\psi \circ F \circ (\varphi')^{-1}\colon \varphi'(U') \to \psi(V)$ is smooth.

(b) We prove the two directions:

($\Rightarrow$) Suppose that F is smooth. By *Proposition 2.4* it is continuous. Now, let (U, φ), respectively (V, ψ), be any smooth chart of M, respectively N. We would like to show that the map $\widehat{F} \coloneqq \psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U \cap F^{-1}(V))$ to $\psi(V)$. If $U \cap F^{-1}(V)$ is empty, then there is nothing to prove. Otherwise, let $p \in U \cap F^{-1}(V)$ be arbitrary. By smoothness of F, there exist charts (W, η)

containing p and (Z, θ) containing $F(p)$ such that $F(W) \subseteq Z$ and such that $\theta \circ F \circ \eta^{-1}$ is smooth from $\eta(W)$ to $\theta(Z)$. In particular, we have

$$\widehat{F} = \psi \circ (\theta^{-1} \circ \theta) \circ F \circ (\eta^{-1} \circ \eta) \circ \varphi^{-1} = (\psi \circ \theta^{-1}) \circ (\theta \circ F \circ \eta^{-1}) \circ (\eta \circ \varphi^{-1})$$

on the open neighborhood $\varphi(U \cap W \cap F^{-1}(V))$ containing $\varphi(p)$. As this is a composition of smooth functions between open subsets of Euclidean spaces, it follows that the function $\widehat{F}$ is smooth in a neighborhood of $\varphi(p)$. As $p \in U \cap F^{-1}(V)$ was arbitrary, we conclude that $\widehat{F}$ is smooth. Hence the maximal smooth atlases of M and N satisfy (b).

($\Leftarrow$) Let $p \in M$, let $(U_\alpha, \varphi_\alpha)$ be a chart containing p and let (V_β, ψ_β) be a chart containing $F(p)$. By hypothesis, $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is smooth from $\varphi_\alpha(U_\alpha \cap F^{-1}(V_\beta))$ to $\psi_\beta(V_\beta)$. As $p \in M$ was arbitrary and since F is continuous, we infer that (a) is satisfied, and thus F is smooth.

Exercise 2: Let M and N be smooth manifolds and let $F \colon M \to N$ be a map. Prove the following assertions:

(a) If every point $p \in M$ has a neighborhood U such that $F|_U$ is smooth, then F is smooth.

(b) If F is smooth, then its restriction to every open subset of M is smooth.

Solution: Recall that any open subset U of M is considered as an open submanifold of M, endowed with the smooth structure $\overline{\mathcal{A}_U}$ determined by the smooth atlas

$$\mathcal{A}_U := \big\{ (W, \theta) \mid (W, \theta) \text{ is a smooth chart for M such that } W \subseteq U \big\}.$$

(a) Let $p \in M$. By hypothesis there exists an open neighborhood U of p in M such that $F|_U$ is smooth. By definition of smoothness, there are smooth charts $(W, \theta) \in \overline{\mathcal{A}_U}$ containing p and (V, ψ) containing $F(p)$ such that $F|_U(W) \subseteq V$ and $\psi \circ (F|_U) \circ \theta^{-1}$ is smooth from $\theta(W)$ to $\psi(V)$. But then (W, θ) is also a smooth chart for M containing p (with $W \subseteq U$) and $F(W) = F|_U(W) \subseteq V$. Since we also have

$$\psi \circ F \circ \theta^{-1} = \psi \circ (F|_U) \circ \theta^{-1}$$

on $\theta(W)$, we conclude that the former is smooth. As $p \in M$ was arbitrary, we infer that F is smooth.

(b) Let U be an open subset of M and let $p \in U$. By smoothness of F there exist smooth charts (W, θ) for M containing p and (V, ψ) for N containing $F(p)$ such that $F(W) \subseteq V$ and such that $\psi \circ F \circ \theta^{-1}$ is smooth from $\theta(W)$ to $\psi(V)$. Now, set $W' := W \cap U$ and $\theta' := \theta|_{W \cap U}$. Then (W', θ') is a smooth chart for U containing p, and we also have $F|_U(W') \subseteq F(W) \subseteq V$ and

$$\psi \circ (F|_U) \circ (\theta')^{-1} = (\psi \circ F \circ \theta^{-1})|_{\theta'(W')}.$$

Hence, $\psi \circ (F|_U) \circ (\theta')^{-1}$ is smooth from $\theta'(W')$ to $\psi(V)$. As $p \in U$ was arbitrary, we conclude that $F|_U$ is smooth.

Exercise 3: Let M, N and P be smooth manifolds. Prove the following assertions:

(a) If $F\colon M \to N$ is a smooth map, then the coordinate representation of F with respect to every pair of smooth charts for M and N is smooth.

(b) If $c\colon M \to N$ is a constant map, then c is smooth.

(c) The identity map $\mathrm{Id}_M\colon M \to M$ is smooth.

(d) If $U \subseteq M$ is an open submanifold, then the inclusion map $\iota\colon U \hookrightarrow M$ is smooth.

(e) If $F\colon M \to N$ and $G\colon N \to P$ are smooth maps, then the composite $G \circ F\colon M \to P$ is also smooth.

Solution:

(a) Fix $p \in M$. Since F is smooth, there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subseteq V$ and $\psi \circ F \circ \varphi^{-1}\colon \varphi(U) \to \psi(V)$ is smooth. Pick smooth charts (U', φ') containing p and (V', ψ') containing $F(p)$. Then $V \cap V'$ is an open neighborhood of $F(p)$ in N, and since F is continuous by *Proposition 2.4*, $F^{-1}(V \cap V')$ is an open neighborhood of p in M, and thus so is $U'' := U \cap U' \cap F^{-1}(V \cap V')$. Consider now the coordinate representation of F with respect to the smooth charts (U', φ') and (V', ψ') with domain of definition $\varphi'(U'')$ and observe that

$$\begin{aligned}\psi' \circ F \circ (\varphi')^{-1} &= \psi' \circ \big(\psi^{-1} \circ \psi\big) \circ F \circ \big(\varphi^{-1} \circ \varphi\big) \circ (\varphi')^{-1} \\ &= \big(\psi' \circ \psi^{-1}\big) \circ \big(\psi \circ F \circ \varphi^{-1}\big) \circ \big(\varphi \circ (\varphi')^{-1}\big).\end{aligned}$$

Thus, $\psi' \circ F \circ (\varphi')^{-1}$ is smooth on its domain of definition as a composition of smooth maps between open subsets of Euclidean spaces; indeed, $\psi \circ F \circ \varphi^{-1}$ is smooth and both $\psi' \circ \psi^{-1}$ and $\varphi \circ (\varphi')^{-1}$ are diffeomorphisms. This proves the claim.

(b) Since c is constant, there exists a point $q \in N$ such that $c(x) = q$ for all $x \in M$. Fix $p \in M$, pick smooth charts (U, φ) containing p and (V, ψ) containing $q = c(p)$, and observe that $\{q\} = c(U) \subseteq V$. Since the composite map $\psi \circ c \circ \varphi^{-1}\colon \varphi(U) \to \psi(V)$ is clearly a constant map (with value $\psi(q)$) between open subsets of Euclidean spaces, it is certainly smooth. Therefore, the given constant map c is smooth.

(c) The identity map $\mathrm{Id}_M\colon M \to M$ of M has an identity map between open subsets of Euclidean spaces as a coordinate representation, so it is smooth.

(d) Fix $p \in U \subseteq M$. Recall that a smooth chart for U containing p is simply a smooth chart (V, ψ) for M such that $p \in V \subseteq U$, and clearly it holds that $\iota(V) = V$. Since the coordinate representation of ι with respect to such a smooth chart is the identity map $\mathrm{Id}_{\psi(V)}\colon \psi(V) \to \psi(V)$, we deduce that $\iota\colon U \hookrightarrow M$ is smooth.

(e) Fix $p \in M$. Since G is smooth, there exist smooth charts (V, ψ) containing $F(p)$ and (W, θ) containing $G(F(p)) = (G \circ F)(p)$ such that $G(V) \subseteq W$ and the composite map $\theta \circ G \circ \psi^{-1}\colon \psi(V) \to \theta(W)$ is smooth. Since F is continuous by *Proposition 2.4*, $F^{-1}(V)$ is an open neighborhood of p in M, and thus there exists a smooth chart (U, φ) for M such that $p \in U \subseteq F^{-1}(V)$. By (a), the composite map $\psi \circ F \circ \varphi^{-1}\colon \varphi(U) \to \psi(V)$ is smooth, and we also have $(G \circ F)(U) \subseteq G(V) \subseteq W$. Now, observe that

$$\theta \circ (G \circ F) \circ \varphi^{-1} = \big(\theta \circ G \circ \psi^{-1}\big) \circ \big(\psi \circ F \circ \varphi^{-1}\big)\colon \varphi(U) \to \theta(W)$$

is smooth as a composition of smooth maps between open subsets of Euclidean spaces. Hence, the composite map $G \circ F \colon M \to P$ is smooth.

Exercise 4: Let $M_1, \ldots, M_k$ be smooth manifolds. For each $i \in \{1, \ldots, k\}$, let

$$\pi_i \colon \prod_{j=1}^{k} M_j \to M_i$$

be the projection onto the i-th factor.

(a) Show that each π_i is smooth.

(b) Let N be a smooth manifold. Show that a map $F \colon N \to \prod_{j=1}^{k} M_j$ is smooth if and only if each of the component maps $F_i := \pi_i \circ F \colon N \to M_i$ is smooth.

Solution:

(a) Let $p = (p_1, \ldots, p_k) \in M_1 \times \ldots \times M_k =: M$ and $1 \leq i \leq k$ be arbitrary. Let (U_i, φ_i) be a smooth chart containing i. By the construction in *Exercise* 5, *Sheet* 2, the smooth structure of M is generated by products of charts of the individual factors. Hence, if for $j \neq i$ we take some chart (U_j, φ_j) of p_j in M_j and write $U = U_1 \times \ldots \times U_k$ resp. $\varphi = \varphi_1 \times \ldots \times \varphi_k$, we obtain that (U, φ) is a chart of p in M. Note then that $\pi_i(U) \subseteq U_i$, and thus the coordinate representation $\widehat{\pi}_i = \varphi_i \circ \pi_i \circ \varphi^{-1}$ of π_i is a map from $\varphi_1(U_1) \times \ldots \times \varphi_k(U_k)$ to $\varphi_i(U_i)$. Furthermore, it is straightforward to see that for all $(v_1, \ldots, v_k) \in \varphi_1(U_1) \times \ldots \times \varphi_k(U_k) \subseteq \mathbb{R}^n$ (where $n := n_1 + \cdots + n_k$), we have

$$\widehat{\pi}_i(v_i) = \varphi_i \circ \pi_i \circ \varphi^{-1}(v_1, \ldots, v_k) = v_i,$$

and thus $\widehat{\pi}_i$ is the projection to the i-th factor $\varphi_1(U_1) \times \cdots \times \varphi_k(U_k) \to \varphi_i(U_i)$. In particular, it is smooth. As $p \in M$ was arbitrary, we conclude that the definition of smoothness is satisfied by π_i; in other words, π_i it is smooth, as claimed.

(b) Suppose first that $F \to \prod_{j=1}^{k} M_j$ is smooth. Pick $1 \leq i \leq k$. By (a) we know that π_i is smooth, and by *Exercise* 3(e) we know that a composition of smooth maps is smooth. Hence, $F_i = \pi_i \circ F$ is smooth.

Suppose now that each of the component maps $F_i = \pi_i \circ F$ is smooth. Let $q \in N$ and set $F(q) = (p_1, \ldots, p_k)$, so that $p_i = F_i(q)$. By hypothesis, for every $1 \leq i \leq k$ there exist smooth charts (V_i, ψ_i) for N containing q and (U_i, φ_i) for M_i containing p_i such that $F_i(V_i) \subseteq U_i$ and such that $\varphi_i \circ F_i \circ \psi_i^{-1}$ is smooth from $\psi_i(V_i)$ to $\varphi_i(U_i)$. Set $V := V_1 \cap \ldots \cap V_k$ and observe that this is an open neighborhood of q. Now, fix any $1 \leq i \leq k$ and set $\psi = \psi_i|_V$. Note that $F_j(V) \subseteq U_j$ for all $1 \leq j \leq k$, so by *Exercise* 3(a) we obtain that $\varphi_j \circ F_j \circ \psi^{-1}$ is smooth from $\psi(V)$ to $\varphi_j(U_j)$ for all j. Moreover, we have

$$F(V) \subseteq F_1(V_1) \times \ldots \times F_k(V_k) \subseteq U_1 \times \ldots \times U_k.$$

In summary, (V, ψ) is a smooth chart for N containing q and $(U_1 \times \ldots \times U_k, \varphi_1 \times \ldots \times \varphi_k)$ is a smooth chart for $M_1 \times \ldots \times M_k$ containing $F(q)$ such that $F(V) \subseteq U_1 \times \ldots \times U_k$, and the coordinate representation

$$(\varphi_1 \times \ldots \times \varphi_k) \circ F \circ \psi^{-1} = (\varphi_1 \circ F_1 \circ \psi^{-1}) \times \ldots \times (\varphi_k \circ F_k \circ \psi^{-1})$$

is smooth from $\psi(V)$ to $\varphi_1(U_1) \times \ldots \times \varphi_k(U_k)$, because all of its components are smooth. As $q \in N$ was arbitrary, we conclude that F is smooth.

Exercise 5: Let M be a smooth manifold of dimension $n \geq 1$. Show that the vector space $C^\infty(M)$ is infinite-dimensional.

[Hint: Show that if $f_1, \ldots, f_k$ are elements of $C^\infty(M)$ with non-empty disjoint supports, then they are linearly independent.]

Solution: Assume first that there is a countable collection $\mathcal{F}$ of smooth functions on M with non-empty disjoint supports. Pick an integer $k \geq 1$. We will show that any k elements $f_1, \ldots, f_k \in \mathcal{F}$ are linearly independent. To this end, write

$$\lambda_1 f_1 + \ldots + \lambda_k f_k = 0 \tag{1}$$

for some $\lambda_i \in \mathbb{R}$. For each $i \in \{1, \ldots, k\}$, pick $x \in \operatorname{supp}(f_i)$ and note that $f_i(x) \neq 0$, whereas $f_j(x) = 0$ for every $j \in \{1, \ldots, k\} \setminus \{i\}$ by assumption. Thus, by evaluating (1) at the chosen point x, we obtain $\lambda_i f_i(x) = 0$, which implies $\lambda_i = 0$. This shows that $f_1, \ldots, f_k \in \mathcal{F}$ are linearly independent, as claimed.

We will now show that there exists a countable collection of smooth functions on M with non-empty disjoint supports, which in turn implies that the $\mathbb{R}$-vector space $C^\infty(M)$ is infinite-dimensional, as desired. Fix a point $p \in M$ and consider a smooth coordinate chart (U, φ) containing p. In view of *Exercise* 1*, Sheet* 1 and by further shrinking U, we may assume that U is a *smooth coordinate cube*, i.e., $\varphi(U) = (0,1) \times \ldots \times (0,1) \subseteq \mathbb{R}^n$. For each integer $i \geq 1$, consider the open subset

$$B_i := (0,1) \times \ldots \times (0,1) \times \left(\frac{1}{i+1}, \frac{1}{i}\right) \subseteq \varphi(U)$$

and pick any non-empty closed subset A_i of B_i. Since $\varphi \colon U \to \varphi(U)$ is a homeomorphism, by *Proposition 2.14* for every $i \geq 1$ there exists a smooth bump function $f_i \in C^\infty(M)$ for $\varphi^{-1}(A_i)$ supported in $\varphi^{-1}(B_i)$. Since $B_i \cap B_j = \emptyset$ whenever $i \neq j$, we also have

$$\operatorname{supp}(f_i) \cap \operatorname{supp}(f_j) = \emptyset \quad \text{for } i \neq j.$$

Therefore, the family $(f_i)_{i=1}^\infty$ is a countable collection of smooth functions on M with non-empty disjoint supports. This completes the proof of the above assertion.

Exercise 6: Let A and B be disjoint closed subsets of a smooth manifold M. Show that there exists $f \in C^\infty(M)$ such that $0 \leq f(x) \leq 1$ for all $x \in M$, $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

Solution: By *Theorem 2.16* there exist non-negative smooth functions f_A and f_B on M such that

$$f_A^{-1}(0) = A \quad \text{and} \quad f_B^{-1}(0) = B. \tag{2}$$

Consider now the function

$$f \colon M \to \mathbb{R},\ x \mapsto \frac{f_A(x)}{f_A(x) + f_B(x)}$$

and observe that it is well-defined (that is, $f_A(x) + f_B(x) \neq 0$ for all $x \in M$) due to (2) and since $A \cap B = \emptyset$. Moreover, f is smooth as a quotient of smooth functions, and it satisfies

$$0 \leq f(x) \leq 1 \quad \text{for all } x \in M,$$

since f_A and f_B are non-negative. Finally, it follows from (2) that

$$f^{-1}(0) = A \quad \text{and} \quad f^{-1}(1) = B.$$

Hence, $f \in C^\infty(M)$ has the desired properties.