choices. From now on we identify $T_p U$ with $T_p M$ for any $p \in U$. This identification just amounts to the observation that $d\iota_p(v)$ is the same derivation as v, thought of as acting on fncts on the bigger mnfd M instead of on fncts on U. Since the action of a derivation on a fnct depends only on the values of the fnct in an arbitrarily small neighborhood, this is a harmless identification. In particular, this means that any tangent vector $v \in T_p M$ can be unambiguously applied to fncts defined only in a neighborhood of p, not necessarily on all of M.

· PROP. 3.10: If M is a smooth n-mnfd, then for each $p \in M$, the tangent space $T_p M$ is an n-dim $\mathbb{R}$-v.s.

PROOF: Fix $p \in M$ and let (U, φ) be a smooth coordinate chart containing p. Since $\varphi : U \longrightarrow \hat{U} \subseteq \mathbb{R}^n$ is a diffeomorphism, $d\varphi_p : T_p U \longrightarrow T_{\varphi(p)} \hat{U}$ is an isomorphism by PROP. 3.7(d). Since PROP. 3.9 guarantees that $T_p U \cong T_p M$ and $T_{\varphi(p)} \hat{U} \cong T_{\varphi(p)} \mathbb{R}^n$, it follows from PROP. 3.3(c) that $\dim T_p M = \dim T_{\varphi(p)} \mathbb{R}^n = n$. ∎

⇝ the tangent space to a vector space : ES4
⇝ the tangent space to a product mnfd : ES4

Next, we will show how to do computations with tangent vectors and differentials in local coordinates.

Let M be a smooth mnfd and let (U, φ) be a smooth coordinate chart on M. Then φ is a diffeomorphism from U to an open subset $\hat{U} \subseteq \mathbb{R}^n$. By PROPs 3.7(d) and 3.9 we infer that $d\varphi_p : T_p M \longrightarrow T_{\varphi(p)} \mathbb{R}^n$ is an isomorphism (for each $p \in U$).

By PROP. 3.3(c) the derivations

$$\frac{\partial}{\partial x^1}\Big|_{\varphi(p)}, \dots, \frac{\partial}{\partial x^n}\Big|_{\varphi(p)}$$

form a basis of $T_{\varphi(p)}\mathbb{R}^n$. Therefore, the preimages of these vectors under the isomorphism $d\varphi_p$, denoted by $\frac{\partial}{\partial x^i}\big|_p$ and characterized by

$$\frac{\partial}{\partial x^i}\Big|_p := (d\varphi_p)^{-1}\left(\frac{\partial}{\partial x^i}\Big|_{\varphi(p)}\right) = d(\varphi^{-1})_{\varphi(p)}\left(\frac{\partial}{\partial x^i}\Big|_{\varphi(p)}\right), \qquad (*_1)$$

form a basis of T_pM. Unwinding the dfns, we see that $\frac{\partial}{\partial x^i}\big|_p$ acts on a fnct $f \in C^\infty(U)$ by

$$\frac{\partial}{\partial x^i}\Big|_p f = \frac{\partial}{\partial x^i}\Big|_{\varphi(p)}(f \circ \varphi^{-1}) = \frac{\partial \hat{f}}{\partial x^i}(\hat{p}),$$

where $\hat{f} := f \circ \varphi^{-1}$ is the coordinate representation of f and $\hat{p} = (p^1, \dots, p^n) = \varphi(p)$ is the coordinate representation of p. In other words, $\frac{\partial}{\partial x^i}\big|_p$ is the derivation that takes the i-th partial derivative of (the coordinate representation of) f at (the coordinate representation of) p. The vectors $\frac{\partial}{\partial x^i}\big|_p$ are called <u>the coordinate vectors at p</u> associated with the given coordinate system.

In summary, if M is a smooth n-mnfd and if $p \in M$, then T_pM is an n-dim $\mathbb{R}$-v.s., and for any smooth chart $(U, (x^i))$ containing p, the coordinate vectors $\left\{\frac{\partial}{\partial x^i}\big|_p\right\}_{i=1}^n$ form a basis for T_pM.

Thus, a tangent vector $v \in T_pM$ can be written uniquely as a linear combination $v = v^i \frac{\partial}{\partial x^i}\big|_p$. The ordered basis $\left(\frac{\partial}{\partial x^i}\big|_p\right)$ is called a <u>coordinate basis for T_pM</u>, and the numbers (v^i) are called the <u>components of v</u> w.r.t. the coordinate basis. If v is known, then its components can be easily computed from its action on the coordinate fncts. For each j, the components of v are given by $v^j = v(x^j)$ (where we think of x^j as a smooth real-valued fnct on U), because

$$v(x^j) = \left(v^i \frac{\partial}{\partial x^i}\bigg|_p \right)(x^j) = v^i \frac{\partial x^j}{\partial x^i}(p) = v^j .$$

We now explore how differentials look in coordinates. We begin by considering the case of a smooth map $F: U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$. For any $p \in U$ we will determine the matrix of $dF_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ in terms of the standard coordinate bases. Denoting by $(x^1, \dots, x^n)$ (resp. $(y^1, \dots, y^m)$) the coordinates in the domain (resp. codomain), we use the chain rule to compute the action of dF_p on a typical basis vector as follows:

$$dF_p\left(\frac{\partial}{\partial x^i}\bigg|_p \right) f = \frac{\partial}{\partial x^i}\bigg|_p (f \circ F) = \frac{\partial f}{\partial y^j}(F(p)) \frac{\partial F^j}{\partial x^i}(p)$$

$$= \left(\frac{\partial F^j}{\partial x^i}(p) \frac{\partial}{\partial y^j}\bigg|_{F(p)} \right) f$$

$$\Rightarrow \quad dF_p\left(\frac{\partial}{\partial x^i}\bigg|_p \right) = \frac{\partial F^j}{\partial x^i}(p) \frac{\partial}{\partial y^j}\bigg|_{F(p)} . \qquad (*_2)$$

In other words, the matrix of dF_p in terms of the coordinates bases is

$$\begin{pmatrix} \frac{\partial F^1}{\partial x^1}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p) \\ \vdots & & \vdots \\ \frac{\partial F^m}{\partial x^1}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p) \end{pmatrix}$$

that is, the Jacobian matrix of F at p, which is the matrix representation of the total derivative $DF(p): \mathbb{R}^n \longrightarrow \mathbb{R}^m$. Therefore, in this case, $dF_p : T_p\mathbb{R}^n \longrightarrow T_{F(p)}\mathbb{R}^m$ corresponds to the total derivative $DF(p): \mathbb{R}^n \longrightarrow \mathbb{R}^m$, under the usual identification of Euclidean spaces with their tangent spaces.

We now consider the more general case of a smooth map $F: M \longrightarrow N$ between smooth mnfds. Choosing smooth coordinate charts (U,φ) for M containing p and (V,ψ) for N containing $F(p)$, we obtain the coordinate representation $\hat{F} = \psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \longrightarrow \psi(V)$, and we also denote by $\hat{p} = \varphi(p)$ the coordinate representation of p. By the computation above, $d\hat{F}_{\hat{p}}$ is represented w.r.t. the standard coordinate bases by the Jacobian matrix of $\hat{F}$ at $\hat{p}$. Using the fact that $F \circ \varphi^{-1} = \psi^{-1} \circ \hat{F}$, we compute

$$dF_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right) \overset{dfn}{=} dF_p\left(d(\varphi^{-1})_{\hat{p}}\left(\frac{\partial}{\partial x^i}\bigg|_{\hat{p}}\right)\right) \underset{3.7(b)}{\overset{\text{PROP.}}{=}}$$

$$= d(\underbrace{F \circ \varphi^{-1}}_{\psi^{-1}\circ\hat{F}})_{\hat{p}}\left(\frac{\partial}{\partial x^i}\bigg|_{\hat{p}}\right) \underset{-\text{"}-}{\overset{-\text{"}-}{=}}$$

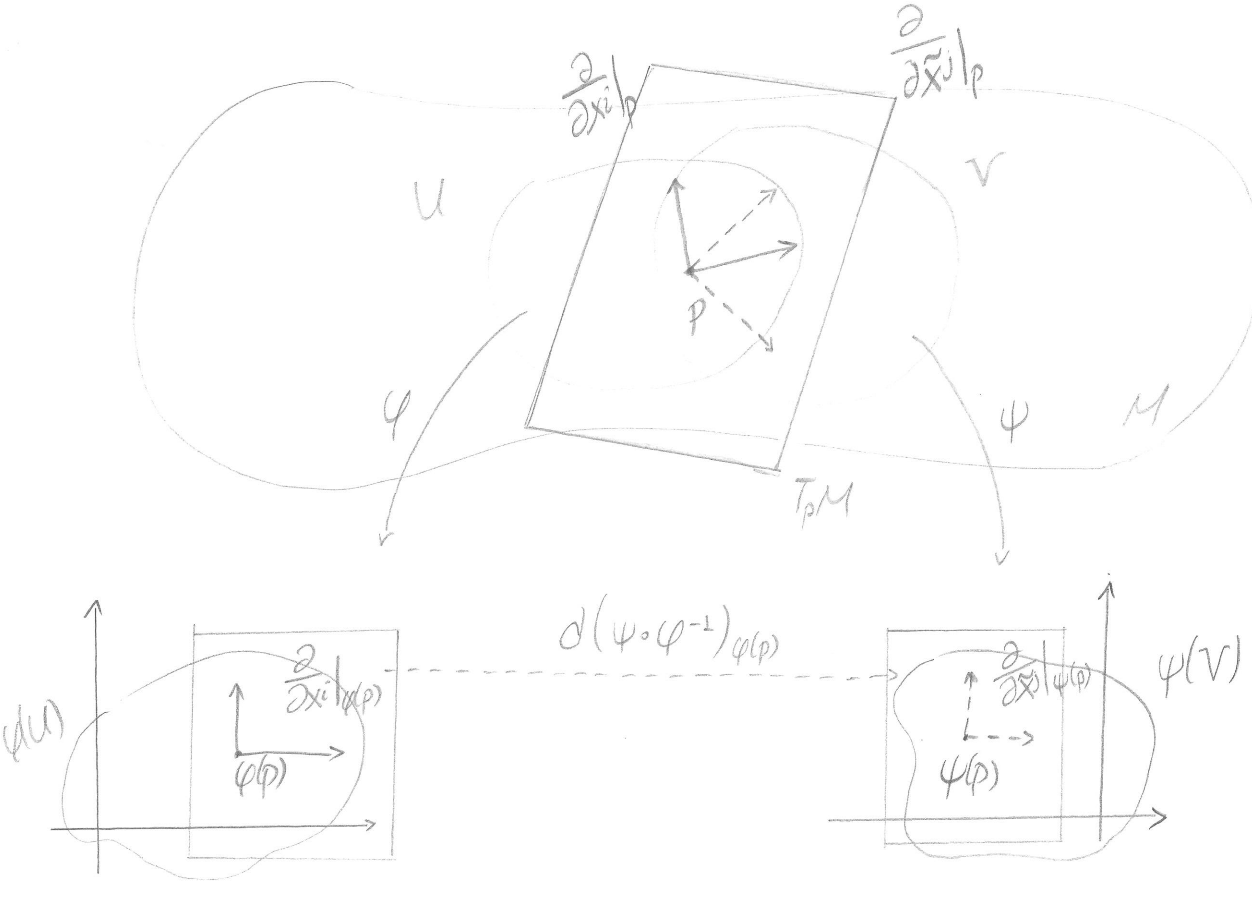
$$= d(\psi^{-1})_{\hat{F}(\hat{p})}\left(d\hat{F}_{\hat{p}}\left(\frac{\partial}{\partial x^i}\bigg|_{\hat{p}}\right)\right) \overset{(*_2)}{=}$$

$$= d(\psi^{-1})_{\hat{F}(\hat{p})}\left(\frac{\partial \hat{F}^j}{\partial x^i}(\hat{p})\frac{\partial}{\partial y^j}\bigg|_{\hat{F}(\hat{p})}\right) \underset{lin}{\overset{dfn}{=}}$$

$$= \frac{\partial \hat{F}^j}{\partial x^i}(\hat{p}) \frac{\partial}{\partial y^j}\bigg|_{F(p)} \qquad (*_3)$$

Thus, dF_p is represented in coordinate bases by the Jacobian matrix of (the coordinate representation $\hat{F}$ of) F. (In fact, the dfn of the differential was cooked up precisely in order to give a coordinate-independent meaning to the Jacobian matrix.)

Finally, suppose that $(U, \varphi = (x^i))$ and $(V, \psi = (\tilde{x}^i))$ are two smooth charts on M, and that $p \in U$. Any tangent vector at p can be represented w.r.t. either coordinates basis $\left(\frac{\partial}{\partial x^i}\big|_p\right)$ or $\left(\frac{\partial}{\partial \tilde{x}^i}\big|_p\right)$. How are the two representations related?



In this situation it is customary to write the transition map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ in the following shorthand notation:

$$\psi \circ \varphi^{-1}(x) = \left(\tilde{x}^1(x), \ldots, \tilde{x}^n(x)\right)$$

Here we are indulging in a typical abuse of notation: in the expression $\tilde{x}^i(x)$, we think of $\tilde{x}^i$ as a coordinate fnct (whose domain is an open subset of M, identified with an open subset of $\mathbb{R}^n$), but we think of x as representing a pt (in this case, in $\varphi(U \cap V)$). By $(*_2)$ we have

$$d(\psi \circ \varphi^{-1})_{\varphi(p)} \left(\frac{\partial}{\partial x^i}\bigg|_{\varphi(p)} \right) = \frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p)) \frac{\partial}{\partial \tilde{x}^j}\bigg|_{\psi(p)}$$

Using the dfn of coordinate vectors, we obtain

$$\frac{\partial}{\partial x^i}\bigg|_p \overset{(*_1)}{=} d(\varphi^{-1})_{\varphi(p)} \left(\frac{\partial}{\partial x^i}\bigg|_{\varphi(p)} \right) \underset{3.7(b)}{\overset{\text{PROP.}}{=}}$$

$$= d(\psi^{-1})_{\psi(p)} \circ d(\psi \circ \varphi^{-1})_{\varphi(p)} \left(\frac{\partial}{\partial x^i}\bigg|_{\varphi(p)} \right)$$

$$= d(\psi^{-1})_{\psi(p)} \left(\frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p)) \frac{\partial}{\partial \tilde{x}^j}\bigg|_{\psi(p)} \right) \underset{\text{lin.}}{\overset{(*_1)}{=}}$$

$$= \frac{\partial \tilde{x}^j}{\partial x^i}(\underset{\hat{p}}{\underset{\shortparallel}{\varphi(p)}}) \frac{\partial}{\partial \tilde{x}^j}\bigg|_p \qquad (*_4)$$

(This formula looks like the chain rule for partial derivatives in $\mathbb{R}^n$.) Applying this to the components of a vector

$$v = v^i \frac{\partial}{\partial x^i}\bigg|_p = \tilde{v}^j \frac{\partial}{\partial \tilde{x}^j}\bigg|_p ,$$

we find that the components of v transform by the rule

$$\tilde{v}^j = \frac{\partial \tilde{x}^j}{\partial x^i}(\hat{p})\, v^i \qquad (*_5)$$

· DEF. 3.11 : Let M be a smooth mnfd. The tangent bundle of M is denoted by TM and is defined as the disjoint union of the tangent spaces at all pts of M :

$$TM = \bigsqcup_{p\in M} T_pM .$$

We usually write an element of this disjoint union as an ordered pair (p,v) with $p\in M$ and $v\in T_pM$ (we sometimes write v_p for (p,v)). The tangent bundle comes equipped with a natural projection map $\pi : TM \longrightarrow M$, which sends each vector in T_pM to the point p at which it is tangent : $(p,v) \mapsto p$.

For example, when $M = \mathbb{R}^n$, using PROP. 3.3 we see that

$$T(\mathbb{R}^n) = \bigsqcup_{p\in\mathbb{R}^n} T_p\mathbb{R}^n \cong \bigsqcup_{p\in\mathbb{R}^n} \mathbb{R}^n_p = \bigsqcup_{p\in\mathbb{R}^n} \{p\}\times\mathbb{R}^n = \mathbb{R}^n\times\mathbb{R}^n .$$

An element of this Cartesian product can be thought of as representing either the geometric tangent vector v_p or the derivation $D_v|_p$ defined in PROP. 3.3. In general, however, the tangent bundle of a smooth mnfd cannot be identified in a natural way with a Cartesian product, because there is no canonical way to identify tangent spaces at distinct pts with each other.

The next proposition shows that the tangent bundle of a smooth mnfd can be considered as a smooth mnfd in its own right. For its proof we need LEM 1.9 (smooth mnfd chart lemma).

· PROP. 3.12 : For any smooth n-mnfd M, the tangent bundle TM has a natural topology and smooth structure that make it into a smooth $(2n)$-mnfd. W.r.t. this structure, the projection $\pi : TM \to M$ is smooth.

<u>PROOF</u>: We begin by defining the maps that will become our smooth charts. Given any smooth chart (U,φ) for M, observe that $\pi^{-1}(U)$ is the set of all tangent vectors to M at all pts of U. Denote by $(x^1, \dots, x^n)$ the coordinate fncts of φ, and define a map

$$\tilde{\varphi} : \pi^{-1}(U) \longrightarrow \mathbb{R}^{2n}$$

$$\tilde{\varphi}\left(v^i \frac{\partial}{\partial x^i}\Big|_p \right) = \left(x^1(p), \dots, x^n(p), v^1, \dots, v^n \right) \qquad (*_6)$$

Its image is the set $\varphi(U) \times \mathbb{R}^n$, which is an open subset of $\mathbb{R}^{2n}$. It is a bijection onto its image, because its inverse can be explicitly written as

$$\tilde{\varphi}^{-1}(x^1, \dots, x^n, v^1, \dots, v^n) = v^i \frac{\partial}{\partial x^i}\Big|_{\varphi^{-1}(x)}.$$

Now, suppose we are given two smooth charts (U,φ) and (V,ψ) for M, and consider the corresponding "charts" $(\pi^{-1}(U), \tilde{\varphi})$ and $(\pi^{-1}(V), \tilde{\psi})$ for TM. The sets

$$\tilde{\varphi}\left(\pi^{-1}(U) \cap \pi^{-1}(V)\right) = \varphi(U \cap V) \times \mathbb{R}^n$$

and

$$\tilde{\psi}\left(\pi^{-1}(U) \cap \pi^{-1}(V)\right) = \psi(U \cap V) \times \mathbb{R}^n$$

are open in $\mathbb{R}^{2n}$, and the transition map

$$\tilde{\psi} \circ \tilde{\varphi}^{-1} : \varphi(U \cap V) \times \mathbb{R}^n \longrightarrow \psi(U \cap V) \times \mathbb{R}^n$$

can be written explicitly as

$$\tilde{\varphi}\circ\tilde{\varphi}^{-1}(x^1,\dots,x^n,v^1,\dots,v^n) = \tilde{\varphi}\left(v^i \frac{\partial}{\partial x^i}\bigg|_{\varphi^{-1}(x)}\right)$$

$$\overset{(*_5)}{=} \tilde{\varphi}\left(\left(v^i \frac{\partial \tilde{x}^j}{\partial x^i}\right)\frac{\partial}{\partial \tilde{x}^j}\bigg|_{\varphi^{-1}(x)}\right)$$

$$= \left(\tilde{x}^1(x),\dots,\tilde{x}^n(x), \frac{\partial \tilde{x}^1}{\partial x^i}v^i,\dots,\frac{\partial \tilde{x}^n}{\partial x^i}v^i\right),$$

which is clearly smooth.

Choosing a countable cover $\{U_i\}$ of M by smooth coordinate domains, we obtain a countable cover of TM by coordinate domains $\{\pi^{-1}(U_i)\}$ satisfying conditions (i)-(iv) of LEM 1.9. To check the Hausdorff condition (v), just note that any two pts in the same fiber of π lie in one chart, while if (p,v) and (q,w) lie in different fibers, there exist disjoint smooth coordinate domains U and V for M s.t. $p\in U$ and $q\in V$, and then $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are disjoint coordinate neighborhoods containing (p,v) and (q,w), respectively. This completes the proof of the first part of the statement.

Finally, to check that $\pi: TM \longrightarrow M$ is smooth, note that w.r.t. charts (U,φ) for M and $(\pi^{-1}(U),\tilde{\varphi})$ for TM, its coordinate representation $\varphi\circ\pi\circ\tilde{\varphi}^{-1}$ is $\pi(x,v) = x$. ■

The coordinates (x^i, v^i) given by $(*_6)$ are called <u>natural coordinates on TM</u>.

⤳ global differential : ES5

⤳ basic properties of global differential : ES5

· PROP. 3.13: If M is a smooth n-mnfd which can be covered by a single smooth chart (U,φ), then its tangent bundle TM is diffeomorphic to $\varphi(U)\times\mathbb{R}^n$.

PROOF: If (U,φ) is a global smooth chart for M, then φ is, in particular, a diffeomorphism from $U=M$ to an open subset $\hat{U}\subseteq\mathbb{R}^n$. The proof of PROP. 3.12 showed that the natural coordinate chart $\tilde{\varphi}$ is a bijection from TM to $\hat{U}\times\mathbb{R}^n$, and the smooth structure on TM is defined essentially by declaring $\tilde{\varphi}$ to be a diffeomorphism. ∎

COMMENT: In general, the tangent bundle is not globally diffeomorphic (or even homeomorphic) to a product of the mnfd with $\mathbb{R}^n$.