Differential Geometry II - Smooth Manifolds
Winter Term 2023/2024
Lecturer: Dr. N. Tsakanikas
Assistant: L. E. Rösler

Exercise Sheet 6

Exercise 1:

(a) Prove the following assertions:

(i) A composition of smooth submersions is a smooth submersion.

(ii) A composition of smooth immersions is a smooth immersion.

(iii) A composition of smooth embeddings is a smooth embedding.

(b) Show by means of a counterexample that a composition of smooth maps of constant rank need not have constant rank.

Exercise 2 (to be submitted by Friday, 3.11.2023, 20:00):

(a) Let $M_1, \ldots, M_k$ be smooth manifolds, where $k \geq 2$. Show that each of the projection maps $\pi_i \colon M_1 \times \ldots \times M_k \to M_i$ is a smooth submersion.

(b) Let $M_1, \ldots, M_k$ be smooth manifolds, where $k \geq 2$. Choosing arbitrarily points $p_1 \in M_1, \ldots, p_k \in M_k$, for each $1 \leq j \leq k$ consider the map

$$\iota_j \colon M_j \to M_1 \times \ldots \times M_k, \; x \mapsto (p_1, \ldots, p_{j-1}, x, p_{j+1}, \ldots, p_k).$$

Show that each ι_j is a smooth embedding.

(c) Examine whether the following plane curves are smooth immersions:

(i) $\alpha \colon \mathbb{R} \to \mathbb{R}^2, \; t \mapsto (t^3, \, t^2)$.

(ii) $\beta \colon \mathbb{R} \to \mathbb{R}^2, \; t \mapsto (t^3 - 4t, \, t^2 - 4)$.

If so, then examine also whether they are smooth embeddings.

(d) Show that the map

$$G \colon \mathbb{R}^2 \to \mathbb{R}^3, \; (u, v) \mapsto \big((2 + \cos 2\pi u) \cos 2\pi v, \, (2 + \cos 2\pi u) \sin 2\pi v, \, \sin 2\pi u \big)$$

is a smooth immersion.

Exercise 3 (*Inverse function theorem for manifolds*)**:**
Let $F\colon M \to N$ be a smooth map. Show that if $p \in M$ is a point such that the differential dF_p of F at p is invertible, then there exist connected neighborhoods U_0 of p in M and V_0 of $F(p)$ in N such that $F|_{U_0}\colon U_0 \to V_0$ is a diffeomorphism.

[Hint: Reduce to the ordinary inverse function theorem for functions between Euclidean spaces.]

Definition:

(a) Let X and Y be topological spaces. A map $F\colon X \to Y$ is called a *local homeomorphism* if every point $p \in X$ has an open neighborhood U such that $F(U)$ is open in Y and $F|_U\colon U \to F(U)$ is a homeomorphism.

(b) Let M and N be smooth manifolds. A map $F\colon M \to N$ is called a *local diffeomorphism* if every point $p \in M$ has an open neighborhood U such that $F(U)$ is open in N and $F|_U\colon U \to F(U)$ is a diffeomorphism.

Exercise 4 (*Elementary properties of local diffeomorphisms*)**:**
Prove the following assertions:

(a) Every composition of local diffeomorphisms is a local diffeomorphism.

(b) Every finite product of local diffeomorphisms between smooth manifolds is a local diffeomorphism.

(c) Every local diffeomorphism is a local homeomorphism and an open map.

(d) The restriction of a local diffeomorphism to an open submanifold is a local diffeomorphism.

(e) Every diffeomorphism is a local diffeomorphism.

(f) Every bijective local diffeomorphism is a diffeomorphism.

(g) A map between smooth manifolds is a local diffeomorphism if and only if in a neighborhood of each point of its domain, it has a coordinate representation that is a local diffeomorphism.

Exercise 5:
Let M and N be smooth manifolds and let $F\colon M \to N$ be a map. Prove the following assertions:

(a) F is a local diffeomorphism if and only if it is both a smooth immersion and a smooth submersion.

(b) If $\dim M = \dim N$ and if F is either a smooth immersion or a smooth submersion, then it is a local diffeomorphism.

Exercise 6:
Let M, N and P be smooth manifolds, and let $F\colon M \to N$ be a local diffeomorphism. Prove the following assertions:

(a) If $G\colon P \to M$ is continuous, then G is smooth if and only if $F \circ G$ is smooth.

(b) If F is surjective and if $H\colon N \to P$ is any map, then H is smooth if and only if $H \circ F$ is smooth.